

Kersten-Krasil'shchik 耦合 KdV-mKdV 方程的达布-贝克隆变换

高 鹏, 薛玲玲*

(宁波大学 数学与统计学院, 浙江 宁波 315211)

摘要: 从矩阵形式的线性系统出发, 构造了 Kersten-Krasil'shchik 耦合 KdV-mKdV 方程的达布变换和相应的贝克隆变换, 并由一对达布-贝克隆变换的相容性得到了非线性叠加公式.

关键词: Kersten-Krasil'shchik 耦合 KdV-mKdV 方程; 达布变换; 贝克隆变换; 非线性叠加公式

中图分类号: O175

文献标志码: A

文章编号: 1001-5132 (2021) 03-0030-05

Korteweg-de Vries(KdV)方程是孤立子理论中非常重要的方程. 它有 $N=1$ ^[1]和 $N=2$ ^[2]超对称形式, 当中 $N=2$, $a=1$ 超对称 KdV 方程为^[2-6]:

$$\Phi_t = -\Phi_{xxx} + 3(\Phi D_1 D_2 \Phi)_x + 3\Phi^2 \Phi_x, \quad (1)$$

其中, $D_i = \partial / \partial \theta_i + \theta_i (\partial / \partial x)$, $(i=1,2)$ 为超数; $\Phi = \Phi(x, t, \theta_1, \theta_2)$ 是依赖于时间变量 t 、空间变量 x 及 Grassmann 奇变量 θ_1 与 θ_2 的玻色函数. 设 $\Phi = v + \theta_2 \beta$, 其中, $\beta = \beta(x, t, \theta_1)$, $v = v(x, t, \theta_1)$ 分别是费米函数和玻色函数, 式(1)可改写为 $N=1$ 的超对称形式:

$$v_t = (-v_{xx} + 3v(D\beta) + v^3)_x, \\ \beta_t = (-\beta_{xx} - 3v(Dv)_x + 3(D\beta)\beta + 3v^2\beta)_x, \quad (2)$$

其中, $D \equiv D_1$. 令 $\beta = \theta_1 u(x, t)$, $v = w(x, t)$, 式(2)退化为经典方程:

$$w_t = (-w_{xx} + 3wu + w^3)_x, \\ u_t = (-u_{xx} - 3ww_{xx} + 3u^2 + 3w^2u)_x. \quad (3)$$

当 $w=0$ 时, 式(3)约化为 KdV 方程 $u_t = (-u_{xx} + 3u^2)_x$; 当 $u=0$ 时, 式(3)约化为 $w_t = -w_{xxx} + 3w^2w_x$ 和 $(ww_{xx})_x = 0$, 前者是 mKdV 方程. 因此, 人们将方程(3)称为 Kersten-Krasil'shchik 耦合 KdV-mKdV 方程^[7]. 此外, 令 $w = i\phi$, $u = -\phi_x$ ($i^2 = -1$), 式(3)还可约化为 3 阶 Burgers 方程 $\phi_t = -(\phi_{xx} + 3\phi\phi_x + \phi^3)_x$.

耦合 KdV-mKdV 方程(3)的很多性质已被研究. Kersten 和 Krasil'shchik 证明该方程完全可积性, 即存在递推算子 and 无穷多对称^[7]. 随后, Kalkanli 等^[8]

证明它具有 Painlevé 性质和 Lax 对, Hon 等^[9]获得了它的孤子解和双周期波解, Qin 等^[10]得到了它的双线性形式并给出了 N -孤子解. 最近, Rui 等^[11]构造了它的拟周期解, Qasim 等^[12]得到了近似解. 但据我们所知, 它的达布变换和贝克隆变换尚未被研究.

达布变换在孤立子理论和可积系统的研究中起着至关重要的作用, 其可以有效地构造出非线性系统不同形式的精确解^[13-14]. 由达布变换可推导出贝克隆变换, 即从一个已知解出发对贝克隆变换积分求得新解. 依据 Bianchi 的可交换性定理, 由贝克隆变换可以生成非线性叠加公式, 从而只需通过微分和代数计算便可以求出新解^[15]. 本文将从耦合 KdV-mKdV 方程的矩阵形式的线性问题出发, 作规范变换推导达布变换和相应的贝克隆变换, 进而构造非线性叠加公式, 从而为进一步研究式(3)的解和构造式(1)的达布变换和非线性叠加公式提供理论基础.

1 达布变换

方便起见, 引入变量 φ 和 ϕ 使得 $w = i\varphi_x$, $u = -\phi_x$. 由式(3)得到势 KdV-mKdV 方程:

$$\varphi_t = -\varphi_{xxx} - 3\varphi_x \phi_x - \varphi_x^3, \\ \phi_t = -\phi_{xxx} - 3\varphi_x \varphi_{xxx} - 3\phi_x^2 - 3\varphi_x^2 \phi_x. \quad (4)$$

由式(2)的 Lax 对^[6]可以得到式(4)的 Lax 对:

$$f_x + \varphi_x g - \lambda f = 0,$$

收稿日期: 2021-01-20.

宁波大学学报(理工版)网址: <http://journal.g.nbu.edu.cn/>

基金项目: 国家自然科学基金(11501312); 宁波大学王宽诚幸福基金.

第一作者: 高鹏(1996-), 男, 江苏南通人, 在读硕士研究生, 主要研究方向: 孤立子理论与可积系统. E-mail: 593885947@qq.com

*通信作者: 薛玲玲(1986-), 女, 浙江宁波人, 副教授, 主要研究方向: 孤立子理论与可积系统. E-mail: xuelingling@nbu.edu.cn

$$\begin{aligned}
&\varphi_x f_x + g_{xx} - \lambda g_x + g\phi_x = 0, \\
&g_t - 3\lambda\phi_{xx}g - 3\lambda\varphi_{xx}f + g_{xxx} + 3\varphi_{xx}f_x + \\
&\quad 3\phi_{xx}g_x - 3\lambda g_{xx} + 3\lambda^2 g_x = 0, \\
&f_t - 3\lambda\phi_{xx}f - 3\lambda\varphi_{xx}g + f_{xxx} + 3\varphi_{xx}g_x + \\
&\quad 3\phi_{xx}f_x - 3\lambda f_{xx} + 3\lambda^2 f_x = 0,
\end{aligned} \quad (5)$$

其中, λ 为谱参数. 引入变量 h 使得 $h_x = g\phi_x + \varphi_x f_x$, 并令 $\Psi = [f, g, h]^T$, 式(5)可改写为矩阵形式:

$$\begin{aligned}
\Psi_x &= M\Psi, \\
M &= \begin{bmatrix} \lambda & -\varphi_x & 0 \\ 0 & \lambda & -1 \\ \lambda\varphi_x & \phi_x - \varphi_x^2 & 0 \end{bmatrix}, \\
\Psi_t &= N\Psi, \\
N &= \begin{bmatrix} -\frac{1}{3}\lambda^3 + i\lambda\varphi_x & \varphi_x^3 + 2\phi_x\varphi_x + \varphi_{xxx} \\ \varphi_{xx} - \lambda\varphi_x & -\frac{1}{3}\lambda^3 + \varphi_x\varphi_{xx} - \lambda\phi_x + \phi_{xx} \\ \gamma & \kappa \end{bmatrix} \rightarrow \\
&\quad \begin{bmatrix} \lambda\varphi_x + \varphi_{xx} \\ \leftarrow \lambda^2 + \varphi_x^2 + 2\phi_x \\ \varsigma \end{bmatrix},
\end{aligned} \quad (6)$$

其中,

$$\begin{aligned}
\gamma &= -\lambda^3\varphi_x + \lambda^2\varphi_{xx} - \lambda\varphi_{xxx} - \lambda\varphi_x^3 - 2\lambda\phi_x\varphi_x, \\
\varsigma &= \frac{2}{3}\lambda^3 + \lambda\phi_x + \lambda\varphi_x^2 - \varphi_x\varphi_{xx} - \phi_{xx}, \\
\kappa &= \varphi_x^4 + \lambda\varphi_x\varphi_{xx} + \phi_x\varphi_x^2 - \lambda^2\phi_x - \varphi_x\varphi_{xxx} - \\
&\quad \varphi_{xx}^2 + \lambda\phi_{xx} - 2\phi_x^2 - \phi_{xxx}.
\end{aligned}$$

可以验证式(6)的相容性条件, 即零曲率方程

$$M_t - N_x + MN - NM = 0,$$

等价于式(3). 注意到, 通过规范变换^[13]:

$$\Psi = e^{\frac{2}{3}\lambda x} \begin{bmatrix} 0 & 0 & \lambda^{-1} \\ 0 & -1 & 0 \\ 1 & 3u & 0 \end{bmatrix} \chi,$$

式(6)等价于文献[10]给出的 Lax 对.

接下来, 构造线性系统(6)的达布变换. 假设存在一个规范变换:

$$\tilde{\Psi} = T\Psi, \quad (7)$$

使得 $\tilde{\Psi}$ 满足:

$$\tilde{\Psi}_x = \tilde{M}\tilde{\Psi}, \tilde{\Psi}_t = \tilde{N}\tilde{\Psi}, \quad (8)$$

其中, 矩阵 $\tilde{M}$ 与 $\tilde{N}$ 是将矩阵 M 与 N 中的场 ϕ 、 φ 分别用新场 $\tilde{\phi}$ 、 $\tilde{\varphi}$ 来替代. 则由式(6)、(7)与(8)的相容性条件, 得到

$$T_x + TM - \tilde{M}T = 0, T_t + TN - \tilde{N}T = 0. \quad (9)$$

假设待定矩阵 T 关于 λ 是一次的, 即 $T = \lambda H + G$, 其中, H 和 G 是与 λ 无关的 3×3 矩阵, 并且将 M 改写为:

$$\begin{aligned}
M &= \lambda U + V, \\
U &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_x & 0 & 0 \end{bmatrix}, \\
V &= \begin{bmatrix} 0 & -\varphi_x & 0 \\ 0 & 0 & -1 \\ 0 & \phi_x - \varphi_x^2 & 0 \end{bmatrix}.
\end{aligned}$$

比较式(9)第1个式子中 λ 的各次幂系数, 得到矩阵方程:

$$\begin{aligned}
O(\lambda^2): HU - \tilde{U}H &= 0, \\
O(\lambda): H_x + HV + GU - \tilde{U}G - \tilde{V}H &= 0, \\
O(\lambda^0): G_x + GV - \tilde{V}G &= 0.
\end{aligned} \quad (10)$$

由式(10)可得:

$$\begin{aligned}
H &= \begin{bmatrix} -c & s & 0 \\ h_{21} & h_{22} & 0 \\ -c\tilde{\varphi}_x - h_{33}\varphi_x & s\tilde{\varphi}_x & h_{33} \end{bmatrix}, \\
G &= \begin{bmatrix} g_{11} & g_{12} & -s \\ g_{21} & g_{22} & h_{33} - h_{22} \\ -g_{21,x} & g_{32} & g_{22} - (h_{33} - h_{22})_x \end{bmatrix},
\end{aligned}$$

其中, c 、 s 、 h_{ij} 和 g_{ij} 是待定函数, 且满足以下超定方程:

$$h_{33,x} = 0, \quad (11)$$

$$g_{12} = -s_x + (h_{33} - h_{22})\tilde{\varphi}_x, \quad (12)$$

$$g_{32} = g_{22,x} - (h_{33} - h_{22})_{xx} + (h_{33} - h_{22}) \cdot (\tilde{\varphi}_x^2 - \tilde{\phi}_x), \quad (13)$$

$$g_{11,x} + g_{21}\tilde{\varphi}_x = 0, \quad (14)$$

$$g_{21}(\tilde{\varphi}_x^2 - \tilde{\phi}_x) - g_{21,xx} = 0, \quad (15)$$

$$c_x + s\varphi_x - h_{21}\tilde{\varphi}_x = 0, \quad (16)$$

$$s_x + c\varphi_x + h_{22}\tilde{\varphi}_x = 0, \quad (17)$$

$$h_{21,x} - c\tilde{\varphi}_x - h_{22}\varphi_x = 0, \quad (18)$$

$$s(\tilde{\varphi} - \varphi)_x + h_{22,x} = 0, \quad (19)$$

$$\begin{aligned}
&(g_{22} - (h_{33} - h_{22})_x)\varphi_x - g_{11}\tilde{\varphi}_x - (c\tilde{\varphi}_x + h_{33}\varphi_x + \\
&\quad g_{21})_x + h_{21}(\tilde{\varphi}_x^2 - \tilde{\phi}_x) = 0,
\end{aligned} \quad (20)$$

$$(h_{33} - h_{22})(\phi_x - \varphi_x^2) - g_{21}\varphi_x + g_{22,x} + g_{32} = 0, \quad (21)$$

$$g_{32,x} + g_{21,x}\varphi_x - (g_{22} - h_{33,x} + h_{22,x})(\varphi_x^2 - \phi_x) + g_{22}(\tilde{\varphi}_x^2 - \tilde{\phi}_x) = 0, \quad (22)$$

$$(s\tilde{\varphi}_x)_x + (c\tilde{\varphi}_x + h_{33}\varphi_x)\varphi_x + g_{32} - h_{33}(\varphi_x^2 - \phi_x) - g_{12}\tilde{\varphi}_x + h_{22}(\tilde{\varphi}_x^2 - \tilde{\phi}_x) = 0. \quad (23)$$

由式(11)知 h_{33} 为常数. 不失一般性, 取 $h_{33} = 1$.

为得到简洁的达布矩阵, 取式(15)的特解, 即 $g_{21} = 0$. 再由式(14)可得 $g_{11} = k_1$, 其中, k_1 是积分常数.

将式(16)和式(19)相加, 式(17)和式(18)相减, 分别得到:

$$(s - h_{21})\tilde{\varphi}_x + (h_{22} + c)_x = 0,$$

$$(h_{22} + c)(\tilde{\varphi}_x + \varphi_x) + (s - h_{21})_x = 0,$$

取其特解:

$$h_{21} = s, h_{22} = -c.$$

此时式(16)和式(17)可分别化为:

$$c_x = s(\tilde{\varphi} - \varphi)_x, s_x = c(\tilde{\varphi} - \varphi)_x. \quad (24)$$

引入辅助变量 r , 使得

$$\tilde{\varphi} = \varphi + \ln(r), \quad (25)$$

并取式(24)的特解:

$$c = \frac{r + r^{-1}}{2}, s = \frac{r - r^{-1}}{2}. \quad (26)$$

下面将 r 和其他未知变量用线性系统(6)的解来表示. 取 $\Psi_1 = [f_1, g_1, h_1]^T$ 是线性系统(6)当 $\lambda = \lambda_1$ 时的解. 特别地, 由式(6)的第 1 个矩阵方程得:

$$\begin{aligned} f_{1,x} &= -\varphi_x g_1 + \lambda_1 f_1, \\ g_{1,x} &= \lambda_1 g_1 - h_1, \\ h_{1,x} &= \lambda_1 \varphi_x f_1 + (\phi_x - \varphi_x^2)g_1. \end{aligned} \quad (27)$$

由附加条件 $T\Psi|_{\lambda=\lambda_1} = \mathbf{0}$ 得到以下方程:

$$(k_1 - \lambda_1 c)f_1 + (\lambda_1 s + g_{12})g_1 - sh_1 = 0, \quad (28)$$

$$\lambda_1 s f_1 + (g_{22} - \lambda_1 c)g_1 + (1 + c)h_1 = 0, \quad (29)$$

$$\begin{aligned} -\lambda_1 f_1(c\tilde{\varphi}_x + \varphi_x) + (\lambda_1 s\tilde{\varphi}_x + g_{32})g_1 + \\ (\lambda_1 + g_{22} + h_{22,x})h_1 = 0. \end{aligned} \quad (30)$$

先求解式(28).

将式(12)整理为:

$$g_{12} = (1 + c)\varphi_x + \frac{r_x}{r}.$$

为了从式(28)中求出 r , 拟设 $r = r(f_1, g_1)$. 利用式(27)的前 2 个式子, g_{12} 的表达式可改写为:

$$g_{12} = (1 + c)\varphi_x + \frac{1}{r} \left(\frac{\partial r}{\partial f_1} (\lambda_1 f_1 - \varphi_x g_1) + \right.$$

$$\left. \frac{\partial r}{\partial g_1} (\lambda_1 g_1 - h_1) \right). \quad (31)$$

将式(31)代入式(28), 并比较 φ_x 和 h_1 的系数, 分别得到以下方程:

$$(1 + c) - \frac{g_1}{r} \left(\frac{\partial r}{\partial f_1} \right) = 0,$$

$$\frac{g_1}{r} \left(\frac{\partial r}{\partial g_1} \right) + s = 0. \quad (32)$$

式(32)的解为:

$$r = \frac{g_1 + f_1 + k_2}{g_1 - f_1 - k_2}, \quad (33)$$

其中, k_2 是积分常数. 将式(33)代入式(28), 可得 $k_1 = -\lambda_1$.

然后, 求解式(29)和式(30), 分别得到:

$$g_{22} = \lambda_1 + \frac{2g_1 h_1 - 2k_2 \lambda_1 (f_1 + k_2)}{(f_1 + g_1 + k_2)(f_1 - g_1 + k_2)}, \quad (34)$$

$$\begin{aligned} g_{32} &= \frac{\lambda_1 f_1}{g_1} (c\tilde{\varphi}_x + \varphi_x) - \frac{h_1}{g_1} (g_{22} - c_x + \\ &\lambda_1) - \lambda_1 s\tilde{\varphi}_x. \end{aligned} \quad (35)$$

利用式(25)、(26)、(33)和(34), 比较式(13)和式(35)可得:

$$\tilde{\phi}_x = \phi_x + \left(\ln((f_1 + g_1 + k_2)(f_1 - g_1 + k_2)) \right)_{xx}.$$

最后, 由式(20)可得 $k_2 = 0$. 可以验证式(21)~(23)成立, 式(9)的第 2 个矩阵方程也成立. 我们将结果整理为以下定理.

定理 1 设 $\Psi_1 = [f_1, g_1, h_1]^T$ 是线性系统(6)当 $\lambda = \lambda_1$ 时的解, 定义

$$\tilde{\varphi} = \varphi + \ln r, r \equiv \frac{g_1 + f_1}{g_1 - f_1}, \quad (36)$$

$$\tilde{\phi} = \phi + \rho, \rho \equiv \left(\ln(f_1^2 - g_1^2) \right)_x, \quad (37)$$

$$\begin{aligned} T = \lambda \begin{bmatrix} -c & s & 0 \\ s & -c & 0 \\ -ab & b + \rho - 2\lambda_1 & 1 \end{bmatrix} + \\ \begin{bmatrix} -\lambda_1 & s\Delta & -s \\ 0 & \lambda_1 - (1 + c)\Delta & 1 + c \\ 0 & \lambda(\rho - 2\lambda_1) + b\Delta & \lambda_1 - b \end{bmatrix}, \end{aligned} \quad (38)$$

其中,

$$c = \frac{r^2 + 1}{2r},$$

$$s = \frac{r^2 - 1}{2r},$$

$$a = \frac{r-1}{r+1},$$

$$b = -s\varphi_x + c(2\lambda_1 - \rho),$$

$$\Delta = a\varphi_x + \frac{2\lambda_1 - \rho}{1+c},$$

则式(7)满足式(8). 其中, T 是达布矩阵.

需要说明的是, 为了给下一节做铺垫, 式(38)中的达布矩阵 T 已经用辅助变量 r 和 ρ 来表示, 即线性系统(6)的解以及 φ . 下面我们用工变量 φ 、 ϕ 、 $\tilde{\varphi}$ 、 $\tilde{\phi}$ 来表示达布矩阵 T . 利用式(36)和式(37), 把辅助变量 r 和 ρ 改写成:

$$\begin{aligned} r &= \exp(\tilde{\varphi} - \varphi), \\ \rho &= \tilde{\phi} - \phi, \end{aligned} \quad (39)$$

将其代入(38), 从而有 $T = T(\lambda_1; \varphi, \phi, \tilde{\varphi}, \tilde{\phi})$, 此时式(9)的第1个矩阵方程等价于耦合 KdV-mKdV 方程空间部分的贝克隆变换:

$$\begin{aligned} (\tilde{\varphi} + \varphi)_x &= (2\lambda_1 - (\tilde{\phi} - \phi)) \tanh\left(\frac{\tilde{\varphi} - \varphi}{2}\right), \\ (\tilde{\phi} - \phi)_x &= \varphi_{xx} \sinh(\tilde{\varphi} - \varphi) - \phi_x (\cosh(\tilde{\varphi} - \varphi) + 1) + \\ &\quad \left(\lambda_1 - \left(\frac{\tilde{\phi} - \phi}{2} \right) \right) \left((\tilde{\phi} - \phi) - (\tilde{\varphi} - \varphi)_x \cdot \right. \\ &\quad \left. \tanh\left(\frac{\tilde{\varphi} - \varphi}{2}\right) \right). \end{aligned} \quad (40)$$

同理, 由式(9)的第2个矩阵方程可得耦合 KdV-mKdV 方程时间部分的贝克隆变换:

$$\begin{aligned} r_t &= -r_{xxx} - 3r_x \varphi_x^2 + 3\varphi_x \\ &\quad \left((2r\varphi_{xx} + r_{xx}) \frac{r+1}{r-1} - \frac{2r_x^2}{r-1} - \frac{4\lambda_1 r_x r}{r^2-1} \right) + \\ &\quad 3\varphi_{xx} r_x \frac{r+1}{r-1} + \frac{6r_x}{r^2-1} (rr_{xx} - 2\lambda_1 r_x - r_x^2), \\ \rho_t &= -\frac{r+1}{r(r-1)} (2r\varphi_{xt} + r_{xt}) + \frac{4\varphi_x r_t}{(r-1)^2} + \\ &\quad \frac{r^2 + 2r - 1}{r^2(r-1)^2} r_x r_t. \end{aligned} \quad (41)$$

2 非线性叠加公式

贝克隆变换式(40)和(41)建立了势耦合 KdV-mKdV 方程(4)的新解 $(\tilde{\varphi}, \tilde{\phi})$ 与旧解 (φ, ϕ) 之间的联

系. 从一个已知解出发, 对贝克隆变换进行积分一次就可以得到新解, 但构造大量的新解有一定困难. 由达布-贝克隆变换的可交换性可获得非线性叠加公式, 利用它只需通过微分和代数运算便可以得到新解^[14].

利用达布变换(7)以及式(38)和式(39), 我们考虑一对达布变换:

$$\tilde{\Psi} = T_1 \Psi, \tilde{\Psi} = T_2 \Psi, \quad (42)$$

其中, $T_1 \equiv T(\lambda_1; \varphi, \phi, \tilde{\varphi}, \tilde{\phi})$, $T_2 \equiv T(\lambda_2; \varphi, \phi, \hat{\varphi}, \hat{\phi})$. 根据 Bianchi 可交换性定理, 即式(42)的相容性条件 $\hat{\tilde{\Psi}} = \tilde{\hat{\Psi}}$ 且 $\hat{\tilde{\phi}} = \tilde{\hat{\phi}}$, $\hat{\tilde{\varphi}} = \tilde{\hat{\varphi}}$, 可得:

$$\hat{T}_1 T_2 = \tilde{T}_2 T_1, \quad (43)$$

其中, $\hat{T}_1 \equiv T(\lambda_1; \hat{\varphi}, \hat{\phi}, \hat{\tilde{\varphi}}, \hat{\tilde{\phi}})$, $\tilde{T}_2 \equiv T(\lambda_2; \tilde{\varphi}, \tilde{\phi}, \hat{\tilde{\varphi}}, \hat{\tilde{\phi}})$. 由式(43)得到非线性叠加公式:

$$\begin{aligned} \hat{\tilde{\varphi}} &= \varphi + \ln Q, \\ \hat{\tilde{\phi}} &= \phi + \frac{2\lambda_1(Q\tilde{r} - \hat{r})(Q + \tilde{r}) - \tilde{r}\hat{\rho}(Q-1)(Q + \hat{r})}{Q(\tilde{r} - \hat{r})(\tilde{r} + 1)} - \\ &\quad \frac{2\lambda_2 Q\tilde{r} - \hat{r})(Q + \tilde{r}) - \hat{r}\hat{\rho}(Q-1)(Q + \hat{r})}{Q(\tilde{r} - \hat{r})(\tilde{r} + 1)}, \end{aligned} \quad (44)$$

其中,

$$\begin{aligned} \tilde{r} &\equiv \exp(\tilde{\varphi} - \varphi), \tilde{\rho} \equiv \tilde{\phi} - \phi, \\ \hat{r} &\equiv \exp(\hat{\varphi} - \varphi), \hat{\rho} \equiv \hat{\phi} - \phi, \\ Q &\equiv \frac{A + \lambda_1 \tilde{r}(\hat{r} + 1)(\tilde{r} - 2\hat{r} - 1) + \lambda_2 \hat{r}(\tilde{r} + 1)(\hat{r} - 2\tilde{r} - 1)}{A + \lambda_1(\hat{r} + 1)(\tilde{r}\hat{r} + 2\tilde{r} - \hat{r}) + \lambda_2(\tilde{r} + 1)(\tilde{r}\hat{r} + 2\hat{r} - \tilde{r})}, \\ A &\equiv \varphi_x(\tilde{r} - \hat{r})(\tilde{r} + \hat{r} + \tilde{r}\hat{r} + 1) + \hat{r}\hat{\rho}(\tilde{r} + 1)^2 - \\ &\quad \tilde{r}\tilde{\rho}(\hat{r} + 1)^2. \end{aligned}$$

因此, 对于给定方程(4)的3组解 (φ, ϕ) 、 $(\tilde{\varphi}, \tilde{\phi})$ 和 $(\hat{\varphi}, \hat{\phi})$, 利用式(44)便可以得到新解.

参考文献:

- [1] Manin Y I, Radul A O. A supersymmetric extension of the Kadomtsev-Petviashvili hierarchy[J]. Communications in Mathematical Physics, 1985, 98(1):65-77.
- [2] Laberge C A, Mathieu P. $N=2$ superconformal algebra and integrable $O(2)$ fermionic extensions of the Korteweg-de Vries equation[J]. Physics Letters B, 1988, 215(4):718-722.
- [3] Labelle P, Mathieu P. A new $N=2$ supersymmetric Korteweg-de Vries equation[J]. Journal of Mathematical Physics, 1991, 32(4):923-927.
- [4] Popowicz Z. The Lax formulation of the "new" $N=2$ SUSY KdV equation[J]. Physics Letters A, 1993,

- 174(5/6):411-415.
- [5] Bourque S, Mathieu P. The Painlevé analysis for $N = 2$ super Korteweg-de Vries equations[J]. Journal of Mathematical Physics, 2001, 42(8):3517-3539.
- [6] Zhang M X, Liu Q P, Shen Y L, et al. Bilinear approach to $N = 2$ supersymmetric KdV equations[J]. Science in China Series A: Mathematics, 2009, 52(9):1973-1981.
- [7] Kersten P, Krasil'shchik J. Complete integrability of the coupled KdV-mKdV system[J]. Advanced Studies in Pure Mathematics, 2002, 37:151-171.
- [8] Kalkanli A K, Sakovich S Y, Yurduşen İ. Integrability of Kersten-Krasil'shchik coupled KdV-mKdV equations: Singularity analysis and Lax pair[J]. Journal of Mathematical Physics, 2003, 44(4):1703-1708.
- [9] Hon Y C, Fan E G. Solitary wave and doubly periodic wave solutions for the Kersten-Krasil'shchik coupled KdV-mKdV system[J]. Chaos, Solitons & Fractals, 2004, 19(5):1141-1146.
- [10] Qin Y, Gao Y T, Yu X, et al. Bell polynomial approach and N -soliton solutions for a coupled KdV-mKdV system [J]. Communications in Theoretical Physics, 2012, 58(1): 73-78.
- [11] Rui W, Qi X. Bilinear approach to quasi-periodic wave solutions of the Kersten-Krasil'shchik coupled KdV-mKdV system[J/OL]. [2020-11-01]. <https://doi.org/10.1186/s13661-016-0634-3>.
- [12] Qasim A F, Al-Amr M O. Approximate solution of the Kersten-Krasil'shchik coupled KdV-mKdV system via reduced differential transform method[J]. Eurasian Journal of Science and Engineering, 2018, 4(2):1-9.
- [13] Matveev V B, Salle M A. Darboux Transformations and Solitons: Springer Series in Nonlinear Dynamics[M]. Berlin: Springer, 1991:1-76.
- [14] 谷超豪, 胡和生, 周子翔. 孤立子理论中的达布变换及其几何应用[M]. 上海: 上海科学技术出版社, 1999:1-60.
- [15] Rogers C, Schief W. Bäcklund and Darboux Transformations Geometry and Modern Applications in Soliton Theory[M]. Cambridge: Cambridge University Press, 2002:1-30; 266-296.

Darboux-Bäcklund transformation for the Kersten-Krasil'shchik coupled KdV-mKdV system

GAO Peng, XUE Lingling*

(School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China)

Abstract: Starting from the matrix linear system, a Darboux transformation and the related Bäcklund transformation for the Kersten-Krasil'shchik coupled KdV-mKdV system are conducted. Based on the compatibility of a pair of Darboux-Bäcklund transformations, a nonlinear superposition formula is also obtained.

Key words: Kersten-Krasil'shchik coupled KdV-mKdV system; Darboux transformation; Bäcklund transformation; nonlinear superposition formula

(责任编辑 章践立)