

非交换无色散 Gelfand-Dickey 方程族的附加对称

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摘要: 本文主要研究非交换无色散 Gelfand-Dickey (GD) 方程族的附加对称, 给出它的 Lax 函数及方程, 定义 Orlov-Schulman 函数 M 和附加流. 在 GD 方程族约束情况下, 找出非交换无色散 GD 方程族附加流中的存活流, 同时研究特殊流得到弦方程并证明其形成一个无穷维李代数 $W_{1+\infty}$.

关键词: 附加对称; 非交换无色散 Gelfand-Dickey (GD) 方程族; 存活流; 弦方程

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无色散可积系统(KP、Toda 等)是低维可积场中的重要系统, 主要应用于数学、物理、拓扑场、矩阵模型等领域^[1-2]. 无色散可积系统的构造是将经典可积系统中的算子和李括号替换为相位函数和泊松括号, 可以看成是经典可积系统的一种形变或类似. 类似经典的可积系统, 无色散可积系统具有许多优良的性质, 如 Lax 对、无穷守恒量和对称性^[3-4]等. 近年来, 附加流显式形式的附加对称性^[5-7]引起了广泛关注. 20 世纪 80 年代, Orlov 等^[5]给出了构造无穷多可积非线性演化方程附加对称性的一般方法. 20 世纪 90 年代, Leur^[8]研究了 KdV 型方程族的弦方程和 $W_{1+\infty}$ 约束, Takasaki 等^[1-4]研究了无色散的可积方程族(KP、Toda). 21 世纪以来, Wang^[9]和 Li^[7]分别对非交换 KP 和 GD 方程族的附加对称做了研究, Wang 等^[10]也研究了无色散非交换 KP 方程族的多哈密顿结构. 本文利用类似的方法研究非交换无色散 GD 方程族的附加对称.

1 预备知识

在非交换可积系统的研究中, 玛雅代数^[11-13]发挥着极其重要的作用, 其中玛雅乘积 $\star$ 的定义为:

$$f \star g = f \cdot \exp \left[\sum_{i \in P} \theta_i \left(\frac{\partial}{\partial t_{2i-1}} \frac{\partial}{\partial t_{2i}} - \frac{\partial}{\partial t_{2i}} \frac{\partial}{\partial t_{2i-1}} \right) \right] g =$$

$$fg + \sum_{i \in P} \theta_i \left(\frac{\partial f}{\partial t_{2i-1}} \frac{\partial g}{\partial t_{2i}} - \frac{\partial f}{\partial t_{2i}} \frac{\partial g}{\partial t_{2i-1}} \right) + g(\theta^2), \quad (1)$$

式中: P 表示一组有限整数; $t_{2i-1}, t_{2i} (i \in P)$ 是非交换坐标; θ_i 表示非交换参数; $g(\theta^2)$ 表示高阶项. 当 $\theta_i \rightarrow 0$ 时, 非交换系统将被简化为交换系统.

非交换无色散 GD 方程族的 Lax 函数 $\mathcal{L}$ 定义为:

$$\mathcal{L} = k^N + \sum_{n=2}^N u_n(t) \star k^{N-n}, \quad (2)$$

其中 $u = u(x, t_1, t_2, \dots)$. 非交换无色散 GD 方程族的 Lax 方程为:

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}_\star, \quad n \neq 0 \pmod{N}, \quad (3)$$

式中: $\mathcal{B}_n = (\mathcal{L}^{\frac{n}{N}})_+$, $n \neq 0 \pmod{N}$, “ $(\cdot)_+$ ” 表示一个多项式上的 k 次方是非负的; 泊松括号的定义为

$\{f, g\}_\star = \frac{\partial f}{\partial k} \star \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \star \frac{\partial g}{\partial k}$. 通过计算发现 $\partial_{t_1} \mathcal{L} = \partial_x \mathcal{L}$, 也就是说 t_1 和 x 是等价的. 为更好地理解 Lax 方程, 在此给出两个具体的例子.

$N=2$ 时, $\mathcal{L} = k^2 + u_2$ 是非交换无色散 KdV 方程族的 Lax 函数, 其中包含非交换无色散 KdV 方程

$$\partial_{t_3} u_2 = \frac{3}{2} \partial_x u_2^2.$$

$N=3$ 时, 非交换无色散 Boussinesq 方程族的 Lax 函数是 $\mathcal{L} = k^3 + u_2 k + u_3$, 相应的非交换无色散 Boussinesq 方程为:

$$\begin{cases} u_{2,t_2} = 2\partial_x u_3, \\ u_{3,t_2} = -\frac{1}{3}(u_2^2)_{xx} + \frac{2}{3}\{u_2, u_3\}_{\theta,t}, \end{cases}$$

$$\text{其中 } \{f, g\}_{\theta,t} = \sum_{i \in P} \theta_i \left[\frac{\partial f}{\partial t_{2i-1}} \frac{\partial g}{\partial t_{2i}} - \frac{\partial f}{\partial t_{2i}} \frac{\partial g}{\partial t_{2i-1}} \right].$$

非交换无色散 GD 方程族的 Lax 函数也可以表示为:

$$\mathcal{L} = e^{ad\varphi} \star (k^N), \quad (4)$$

其中 dressing 函数 φ 满足

$$\varphi(t) = \sum_{n=1}^{\infty} \varphi_n(t) k^{-n}, \quad n \neq 0 \pmod{N},$$

$$ad\varphi \star (\psi) = \{\varphi, \psi\}_\star.$$

另外, 式(3)等价于下述 Sato 方程:

$$\nabla_{t_n, \varphi} \varphi = -\left(e^{ad\varphi} \star (k^n)\right)_-. \quad (5)$$

为了方便研究非交换无色散 GD 方程族的附加对称, 引入 Orlov-Schulman 函数 $\mathcal{M}$, 假定函数

$$\Gamma = \sum_{n=1}^{\infty} n t_n k^{n-1}, \quad n \neq 0 \pmod{N},$$

同时将 Orlov-Schulman 函数 $\mathcal{M}$ 定义为:

$$\begin{aligned} \mathcal{M} &= e^{ad\varphi} \star (\Gamma) = e^{ad\varphi} \star e^{adt(k)} \star (x), \\ t(k) &= \sum_{n=1}^{\infty} t_n k^n, \quad n \neq 0 \pmod{N}. \end{aligned} \quad (6)$$

进一步计算, 函数 $\mathcal{M}$ 也可以用 Lax 函数 $\mathcal{L}$ 表示:

$$\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{\frac{n-1}{N}} + x + \sum_{i=1}^{\infty} v_{i+1} \star \mathcal{L}^{\frac{-i-1}{N}}. \quad (7)$$

性质 1 函数 $\mathcal{M}$ 满足下列等式:

$$\partial_{t_n} \mathcal{M} = \{\mathcal{B}_n, \mathcal{M}\}_\star, \quad \{\mathcal{L}^{\frac{1}{N}}, \mathcal{M}\}_\star = 1. \quad (8)$$

证明 先证明第二个结论,

$$\{\mathcal{L}^{\frac{1}{N}}, \mathcal{M}\}_\star = e^{ad\varphi} \star \{k, \Gamma\}_\star = e^{ad\varphi} \star 1 = 1.$$

根据 $\partial(e^{adY} \star X) = e^{adY} \star (\partial X) + \{\nabla_Y Y, e^{adY} \star (X)\}_\star$, 对方程(6)进行计算,

$$\begin{aligned} \partial_{t_n} \mathcal{M} &= \partial_{t_n} (e^{ad\varphi} \star (\Gamma)) = \\ &= e^{ad\varphi} \star (\partial_{t_n} \Gamma) + \{\nabla_{t_n, \varphi} \varphi, e^{ad\varphi} \star (\Gamma)\}_\star = \\ &= \{\nabla_{t_n, \varphi} \varphi, \mathcal{M}\}_\star + e^{ad\varphi} \star (n k^{n-1}) = \\ &= \{\mathcal{B}_n - e^{ad\varphi} \star (k^n), \mathcal{M}\}_\star + e^{ad\varphi} \star (n k^{n-1}) = \\ &= e^{ad\varphi} \star (\{e^{-ad\varphi} \star \mathcal{B}_n - k^n, \Gamma\}_\star + \{k^n, \Gamma\}_\star) = \end{aligned}$$

$$e^{ad\varphi} \star \{e^{-ad\varphi} \star \mathcal{B}_n, \Gamma\}_\star = \{\mathcal{B}_n, \mathcal{M}\}_\star,$$

即为所求, 证毕.

2 附加对称

首先给出非交换无色散 GD 方程族的附加对称.

定义 1 微分方程

$$\nabla_{t_{m,n}, \varphi} \varphi = -(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_- \quad (9)$$

的解被称为非交换无色散 GD 方程族的附加对称, 其中 $\partial_{t_{m,n}}$ 表示依赖于一些附加变量 $t_{m,n}$ 的导数.

有了上述准备, 现在来定义附加流, 并且证明这些流是对称的. 类似于无色散 KP 方程族的情形, 引入附加变量 $\partial_{t_{m,n}}$ 并且定义作用在函数 $\mathcal{L}$ 和 $\mathcal{M}$ 上的附加流为:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_{m,n}} &= -\{(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-, \mathcal{L}\}_\star, \\ \frac{\partial \mathcal{M}}{\partial t_{m,n}} &= -\{(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-, \mathcal{M}\}_\star. \end{aligned} \quad (10)$$

定理 1 在非交换无色散 GD 方程族中, 只有

满足条件 $(\mathcal{M}^{m-1} \star \mathcal{L}^{\frac{N+n-1}{N}})_- = 0$ 的附加流存活.

证明 对于非交换无色散 GD 方程族的 Lax 函数(2)而言, 它的负部等于 0. 根据式(10)有

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_{m,n}} &= -\left\{ \left(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}} \right)_-, \mathcal{L} \right\}_\star = \\ &= -\left(\left(e^{ad\varphi} \star (\Gamma^m) \star e^{ad\varphi} \star (k^n) \right)_-, e^{ad\varphi} \star (k^N) \right)_\star. \end{aligned}$$

经过一系列计算, 上述等式还可以写为:

$$\frac{\partial \mathcal{L}}{\partial t_{m,n}} = mN(\mathcal{M}^{m-1} \star \mathcal{L}^{\frac{N+n-1}{N}})_-,$$

当且仅当 $(\mathcal{M}^{m-1} \star \mathcal{L}^{\frac{N+n-1}{N}})_- = 0$ 时, $\partial_{t_{m,n}} \mathcal{L} = 0$ 成立.

另外, 非交换无色散 GD 方程族中存活的附加流也有一些很重要的性质.

性质 2 在非交换无色散 GD 方程族中, 附加流 $\partial_{t_{m,l}}$ 和原始流 ∂_{t_n} 交换, 即 $[\partial_{t_{m,l}}, \partial_{t_n}]_\star = 0, n \neq 0 \pmod{N}$.

证明 将上式两边作用在 Lax 函数 $\mathcal{L}$ 上, 运用方程(3)和(10)有

$$\begin{aligned} [\partial_{t_{m,l}}, \partial_{t_n}]_\star \mathcal{L} &= \partial_{t_{m,l}} (\partial_{t_n} \mathcal{L}) - \partial_{t_n} (\partial_{t_{m,l}} \mathcal{L}) = \\ &= \partial_{t_{m,l}} \{\mathcal{B}_n, \mathcal{L}\}_\star + \partial_{t_n} \{(\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_-, \mathcal{L}\}_\star = \end{aligned}$$

$$\begin{aligned}
& \partial_{t_{m,l}} \{-(\mathcal{L}^{\frac{n}{N}})_-, \mathcal{L}\}_* - \partial_{t_n} \{(\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+, \mathcal{L}\}_* = \\
& \left\{ - \left(\{(\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+, \mathcal{L}^{\frac{n}{N}}\}_* \right)_-, \mathcal{L} \right\}_* + \\
& \left\{ -(\mathcal{L}^{\frac{n}{N}})_-, \{(\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+, \mathcal{L}\}_* \right\}_* - \\
& \left\{ \left(\{-(\mathcal{L}^{\frac{n}{N}})_-, \mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}}\}_* \right)_+, \mathcal{L} \right\}_* - \\
& \left\{ (\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+, \{-(\mathcal{L}^{\frac{n}{N}})_-, \mathcal{L}\}_* \right\}_* = \\
& \left\{ \{(\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+, -(\mathcal{L}^{\frac{n}{N}})_-\}_*, \mathcal{L} \right\}_* + \\
& \left\{ -(\mathcal{L}^{\frac{n}{N}})_-, \{(\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+, \mathcal{L}\}_* \right\}_* - \\
& \left\{ (\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+, \{-(\mathcal{L}^{\frac{n}{N}})_-, \mathcal{L}\}_* \right\}_* = \\
& \left\{ -(\mathcal{L}^{\frac{n}{N}})_-, \{(\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+, \mathcal{L}\}_* \right\}_* + \\
& \left\{ (\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+, \{\mathcal{L}, -(\mathcal{L}^{\frac{n}{N}})_-\}_* \right\}_* + \\
& \left\{ \mathcal{L}, \{-(\mathcal{L}^{\frac{n}{N}})_-, (\mathcal{M}^m \star \mathcal{L}^{\frac{l}{N}})_+\}_* \right\}_* = 0.
\end{aligned}$$

性质 3 非交换无色散 GD 方程族的附加流 $\partial_{t_{m,n}}$ 可以形成无中心的 $W_{1+\infty}$ 代数.

证明 直接计算可以得到

$$[\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}}, \mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}}]_* = \sum_{ab} C_{ab}^{(nm)(kl)} \mathcal{M}^a \star \mathcal{L}^{\frac{b}{N}},$$

这意味着

$$([\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}}, \mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}}]_-)_- = \sum_{ab} C_{ab}^{(nm)(kl)} (\mathcal{M}^a \star \mathcal{L}^{\frac{b}{N}})_-,$$

式中, 系数 $C_{ab}^{(nm)(kl)}$ 是 W 代数中的标准系数. 结合式(10)有

$$\begin{aligned}
& [\partial_{t_{m,n}}, \partial_{t_{l,k}}]_* \mathcal{L} = \partial_{t_{m,n}} (\partial_{t_{l,k}} \mathcal{L}) - \partial_{t_{l,k}} (\partial_{t_{m,n}} \mathcal{L}) = \\
& -\partial_{t_{m,n}} \{(\mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}})_-, \mathcal{L}\}_* + \\
& \partial_{t_{l,k}} \{(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-, \mathcal{L}\}_* = \\
& \left\{ \left(\{(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-, \mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}}\}_* \right)_-, \mathcal{L} \right\}_* +
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}})_-, \{(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-, \mathcal{L}\}_* \right\}_* - \\
& \left\{ \left(\{(\mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}})_-, \mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}}\}_* \right)_-, \mathcal{L} \right\}_* - \\
& \left\{ (\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-, \{(\mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}})_-, \mathcal{L}\}_* \right\}_* = \\
& \left\{ \left(\{(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-, \mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}}\}_* \right)_-, \mathcal{L} \right\}_* - \\
& \left\{ \left(\{(\mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}})_-, \mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}}\}_* \right)_-, \mathcal{L} \right\}_* + \\
& \left\{ \{(\mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}})_-, (\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-\}_*, \mathcal{L} \right\}_* = \\
& \left\{ \left(\{(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-, \mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}}\}_* \right)_-, \mathcal{L} \right\}_* + \\
& \left\{ \left(\{(\mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}})_-, (\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}})_-\}_* \right)_-, \mathcal{L} \right\}_* = \\
& \left\{ \{(\mathcal{M}^m \star \mathcal{L}^{\frac{n}{N}}, \mathcal{M}^l \star \mathcal{L}^{\frac{k}{N}})_-\}_*, \mathcal{L} \right\}_*,
\end{aligned}$$

也就是说

$$\begin{aligned}
& [\partial_{t_{m,n}}, \partial_{t_{l,k}}]_* \mathcal{L} = \left\{ \left(\sum_{ab} C_{ab}^{(nm)(kl)} \mathcal{M}^a \star \mathcal{L}^{\frac{b}{N}} \right)_-, \mathcal{L} \right\}_* = \\
& -\sum_{ab} C_{ab}^{(nm)(kl)} \partial_{t_{a,b}} \mathcal{L}.
\end{aligned}$$

因此, $[\partial_{t_{m,n}}, \partial_{t_{l,k}}]_* = -\sum_{ab} C_{ab}^{(nm)(kl)} \partial_{t_{a,b}}$.

3 弦方程

本节在非交换无色散 GD 方程族附加对称的基础上, 给出它的弦方程.

回顾等式 $\{\mathcal{L}^{\frac{1}{N}}, \mathcal{M}\}_* = 1$, 变形得 $\{\mathcal{M}, \mathcal{L}^{\frac{n}{N}}\}_* = -n\mathcal{L}^{\frac{n-1}{N}}$. 进一步计算有

$$\{\mathcal{M} \star \mathcal{L}^{\frac{-n+1}{N}}, \mathcal{L}^{\frac{n}{N}}\}_* = -n.$$

取特殊的附加流 $\partial_{t_{1,-(n-1)}}$ 并将其作用在函数 $\mathcal{L}^{\frac{n}{N}}$ 上, 可以得出

$$\begin{aligned}
& \frac{\partial \mathcal{L}^{\frac{n}{N}}}{\partial t_{1,-(n-1)}} = -\{(\mathcal{M} \star \mathcal{L}^{\frac{-n+1}{N}})_-, \mathcal{L}^{\frac{n}{N}}\}_* = \\
& \{(\mathcal{M} \star \mathcal{L}^{\frac{-n+1}{N}})_+, \mathcal{L}^{\frac{n}{N}}\}_* + n.
\end{aligned}$$

假定 $\mathcal{L}^{\frac{n}{N}}$ 是一个多项式函数, 即 $\mathcal{L}^{\frac{n}{N}} = (\mathcal{L}^N)_+$. 因此,

$$\{\mathcal{L}^{\frac{n}{N}}, \frac{1}{n}(\mathcal{M} \star \mathcal{L}^{\frac{-n+1}{N}})_+\}_{\star} = 1$$

就是非交换无色散 GD 方程族的弦方程. 从上述推导中可以看出弦方程是指函数与附加变量无关的条件, 即 $\partial_{t_{1-(n-1)}} \mathcal{L}^{\frac{n}{N}} = 0$.

4 总结

本文主要研究非交换无色散 GD 方程族的附加对称及存活的附加流的性质. 由于该方程族 Lax 函数的特殊性, 只有满足条件 $(\mathcal{M}^{m-1} \star \mathcal{L}^{\frac{N+n-1}{N}})_- = 0$ 的附加流存活, 这是与非交换无色散 KP 方程族的一大不同. 通过特殊附加流 $\partial_{t_{1-(n-1)}}$ 对 $\mathcal{L}^{\frac{n}{N}}$ 作用, 得到弦方程.

参考文献:

- [1] Takasaki K, Takebe T. Integrable hierarchies and dispersionless limit[J]. Reviews in Mathematical Physics, 1995, 7(5):743-808.
- [2] Takasaki K, Takebe T. SDIFF(2) KP hierarchy[J]. International Journal of Modern Physics A, 1993, 7:889-922.
- [3] Takasaki K. Symmetries and tau function of higher dimensional dispersionless integrable hierarchies[J]. Journal of Mathematical Physics, 1995, 36(7):3574-3607.
- [4] Takasaki K. Dispersionless Toda hierarchy and two-dimensional string theory[J]. Communications in Mathematical Physics, 1995, 170(1):101-116.
- [5] Orlov A, Schulman E. Additional symmetries for integrable equations and conformal algebra representation[J]. Letters in Mathematical Physics, 1986, 12:171-179.
- [6] Dickey L A. Soliton equations Hamiltonian systems[M]. 2nd ed. Singapore: World Scientific, 2003.
- [7] Li C Z. Symmetries and reductions on the non-commutative Kadomtsev-Petviashvili and GD hierarchies[J]. Journal of Mathematical Physics, 2018, 51:5-11.
- [8] Leur J V D. KdV type hierarchies, the string equation and $W_{1+\infty}$ constraints[J]. Journal of Geometry and Physics, 1995, 17(2):95-124.
- [9] Wang N. On symmetry flows of noncommutative Kadomtsev-Petviashvili hierarchy[J]. Communications in Theoretical Physics, 2005, 9(44):39-45.
- [10] Wang N, Zhang X W. Dispersionless noncommutative Kadomtsev-Petviashvili hierarchy: Multi-Hamiltonian structure and large N-like V-algebras[J]. Physics Letters A, 2006, 355:188-192.
- [11] Strachan I A B. A geometry for multidimensional integrable systems[J]. Journal of Geometry and Physics, 1997, 21(3):255-278.
- [12] Strachan I A B. The Moyal algebra and integrable deformations of the self-dual Einstein equations[J]. Physics Letters B, 1992, 283(1):63-66.
- [13] Strachan I A B. The Moyal bracket and the dispersionless limit of the KP hierarchy[J]. Journal of Physics A: General Physics, 1994, 28(7):1967-1975.

Additional symmetry of noncommutative dispersionless Gelfand-Dickey hierarchy

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Abstract: This article mainly deals with the additional symmetry of the noncommutative dispersionless Gelfand-Dickey (GD) hierarchy. We give its Lax function and equation, and define the Orlov-Schulman function $\mathcal{M}$ and the additional flows. Under the constraints of the GD hierarchy, we find that only a part of additional flows can survive. Considering the special additional flow, the string equation is obtained. In addition, this part of additional flow forms an infinite-dimensional Lie algebra $W_{1+\infty}$.

Key words: additional symmetry; noncommutative dispersionless Gelfand-Dickey (GD) hierarchy; survival flow; string equation

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