

Determinants and Inverses of r -circulant Matrices with Jacobsthal and Jacobsthal-Lucas Numbers

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Abstract: Let $J_n = C_r(J_1, J_2, \dots, J_n)$ and $j_n = C_r(j_0, j_1, \dots, j_{n-1})$ be r -circulant matrix of order n , where $J_1, J_2, \dots, J_n$ are the Jacobsthal numbers and $j_0, j_1, \dots, j_{n-1}$ are the Jacobsthal-Lucas numbers. We obtained the values of the determinants of matrices J_n and j_n in terms of the Jacobsthal number and the Jacobsthal-Lucas number. Moreover, the inverses of matrices J_n and j_n are also derived.

Key words: r -circulant matrix; determinant; inverse; Jacobsthal number; Jacobsthal-Lucas number

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The $n \times n$ r -circulant matrices $C_n = [c_{ij}] \in M_{n,n}(C)$ is defined as

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i, \\ rc_{n+j-i}, & j < i. \end{cases}$$

Apparently, the r -circulant matrices C_n is determined by parameter r and its first row elements $c_0, c_1, \dots, c_{n-1}$, hence we denote $C_n = C_r(c_0, c_1, \dots, c_{n-1})$. Extraordinary, let $r=1$, the matrix C_n is called a circulant matrix.

The r -circulant matrices is widely applied in various fields, such as in signal processing, image processing, self-regress design, engineering calculation and so on. Thus many scholars have researched it. In the [1-2], the authors obtained a well known algorithms for computing the determinant and inverse of nonsingular circulant matrix C_n . The formulas are given as follows:

$$\det C_n = \prod_{j=0}^{n-1} g(w^j), \quad C_n^{-1} = C_r(b_0, b_1, \dots, b_{n-1}),$$

where

$$b_k = \frac{1}{n} \sum_{j=1}^{n-1} g^{-1}(w^j) w^{-jk} \quad (k=0, 1, \dots, n-1),$$

$$g(x) = \sum_{i=0}^{n-1} a_i x^i,$$

and $w = \exp(2\pi i / n)$ is the root of $x^n = r$.

The computational complexity of these algorithms is not feasible when the order of matrix is getting large. In recent years, it was appeared studies on the circulant matrices with interesting number sequences. We obtained more explicit forms of the determinants and inverses of the corresponding C_n . In [3] Shen give formulas for the determinants and inverses of the circulant matrix $A_n = \text{circ}(F_1, F_2, \dots, F_n)$ and $B_n = (L_1, L_2, \dots, L_n)$. The Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$ are defined by

$$F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0, F_1 = 1,$$

$$L_{n+1} = L_n + L_{n-1}, \quad L_0 = 2, L_1 = 1.$$

In [4], the determinants and inverses of $J_n = \text{circ}(J_1, J_2, \dots, J_n)$ and $j_n = \text{circ}(j_0, j_1, \dots, j_{n-1})$ are obtained only in terms of the Jacobsthal and Jacobsthal-Lucas numbers, respectively. The Jacobsthal $\{J_n\}$ and Jacobsthal-Lucas sequence $\{j_n\}$ are defined by

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, J_1 = 1,$$

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$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 2, j_1 = 1.$$

Let α and β be the roots of the $x^2 - x - 2 = 0$, then the Binet formulas^[5] of the sequences $\{J_n\}$ and $\{j_n\}$ have the form

$$J_n = \frac{\alpha^n - \beta^n}{3} = \frac{1}{3}[2^n - (-1)^n],$$

$$j_n = \alpha^n + \beta^n = 2^n + (-1)^n.$$

The main content of this paper is promotion the determinants and inverses of circulant matrices to r -circulant matrices associated with the Jacobsthal and Jacobsthal-Lucas numbers.

1 Determinant and inverse of r -circulant matrix with the Jacobsthal numbers

In this section, we give a determinant formula for the r -circulant matrix $\mathcal{J}_n = C_r(J_1, J_2, \dots, J_n)$ firstly. Then, we use the fact that the inverse of a nonsingular r -circulant matrix is also r -circulant matrix, and obtained the formula for the inverse of r -circulant matrix $\mathcal{J}_n$.

Theorem 1 Let $n \geq 3$. Then we have

$$\det \mathcal{J}_n = (1 - rJ_{n+1})^{n-1} + 2r \sum_{k=1}^{n-1} J_k (2rJ_n)^{n-k-1}.$$

$$(1 - rJ_{n+1})^{k-1}, \quad (1)$$

where J_n is the n th Jacobsthal number.

So $\mathcal{J}_n$ is nonsingular for all $n \geq 3$.

Proof By direct computation $\det \mathcal{J}_3$ satisfied (1).

In the case $n > 3$, let

$$P_n =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -r & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -2r & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & -2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 1 & -1 & \ddots & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$Q_1 =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \left(\frac{2rJ_n}{1-rJ_{n+1}}\right)^{n-2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \left(\frac{2rJ_n}{1-rJ_{n+1}}\right)^{n-3} & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & \left(\frac{2rJ_n}{1-rJ_{n+1}}\right)^{n-4} & 0 & 0 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \left(\frac{2rJ_n}{1-rJ_{n+1}}\right)^2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \left(\frac{2rJ_n}{1-rJ_{n+1}}\right) & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then we have the following relation:

$$S_n = P_n \mathcal{J}_n Q_1 =$$

$$\begin{bmatrix} J_1 g'_n - J_n & -J_{n-1} & \cdots & -J_5 & -J_4 & -J_3 \\ g_n rJ_n - 1 & -2rJ_{n-2} & \cdots & -2rJ_4 & -2rJ_3 & -2rJ_2 \\ 2rJ_n & rJ_{n+1} - 1 & & & & \\ & 2rJ_n & \ddots & & & 0 \\ & & \ddots & \ddots & & \\ & & & \ddots & rJ_{n+1} - 1 & \\ & & & & 2rJ_n & rJ_{n+1} - 1 \\ & & & & & 2rJ_n \end{bmatrix},$$

which is a upper triangular, where

$$g'_n = \sum_{k=1}^{n-1} J_{k+1} \left(\frac{2rJ_n}{1-rJ_{n+1}}\right)^{n-k-1},$$

and

$$g_n = 1 - rJ_{n+1} + 2r \sum_{k=1}^{n-1} J_k \left(\frac{2rJ_n}{1-rJ_{n+1}}\right)^{n-k-1}.$$

Then we obtained

$$\det S_n = \det P_n \det \mathcal{J}_n \det Q_1 = g_n (2rJ_n)^{n-2},$$

while

$$\det P_n = (-1)^{(n-1)(n-2)/2},$$

$$\text{and } \det Q_1 = \left(\frac{2rJ_n}{1-rJ_{n+1}}\right)^{n-2} (-1)^{\frac{(n-1)(n-2)}{2}}.$$

$$\text{We have } \det P_n \det Q_1 = \left(\frac{2rJ_n}{1-rJ_{n+1}}\right)^{n-2}, \text{ thus}$$

$$\det \mathcal{J}_n = (1 - rJ_{n+1})^{n-1} + 2r \sum_{k=1}^{n-1} J_k (2rJ_n)^{n-k-1} (1 - rJ_{n+1})^{k-1}.$$

Lemma 1 Let the matrix $A = [a_{ij}]_{i,j=1}^{n-2}$ is defined as

$$a_{ij} = \begin{cases} 2rJ_n, & i = j, \\ rJ_{n+1} - 1, & j = i + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then the inverse $A^{-1} = [a'_{ij}]_{i,j=1}^{n-2}$ is equal to

$$a'_{ij} = \begin{cases} \frac{(1-rJ_{n+1})^{j-i}}{(2rJ_n)^{j-i+1}}, & j \geq i, \\ 0, & \text{otherwise.} \end{cases}$$

Proof With the elementary row transformation, we can take $(I - \lambda N : I)$ into $(I : (I - \lambda N)^{-1})$.

Then we obtained the inverse

$$(I - \lambda N)^{-1} = (I + \lambda N + \lambda^2 N^2 + \cdots + \lambda^{n-3} N^{n-3}),$$

where $\lambda \in \mathbb{C}$, I is $(n-2)$ order unite matrix and N is $(n-2)$ order strictly upper triangular nilpotent matrix

$$N = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix},$$

So $A = 2rJ_n \left(I - \frac{1-rJ_{n+1}}{2rJ_n} N \right)$.

we obtained

$$A^{-1} = \frac{1}{2rJ_n} \left(I + \left(\frac{1-rJ_{n+1}}{2rJ_n} \right) N + \left(\frac{1-rJ_{n+1}}{2rJ_n} \right)^2 N^2 + \cdots + \left(\frac{1-rJ_{n+1}}{2rJ_n} \right)^{n-3} N^{n-3} \right).$$

$$Q_2 = \begin{bmatrix} 1 & g'_n & \frac{g'_n}{g_n}(J_n - 1) + J_n & J_{n-1} - \frac{2g'_n}{g_n} \cdot rJ_{n-2} & \cdots & J_4 - \frac{2g'_n}{g_n} \cdot rJ_3 & J_3 - \frac{2g'_n}{g_n} \cdot rJ_2 \\ 0 & 1 & -\frac{J_n - 1}{g_n} & \frac{2rJ_{n-2}}{g_n} & \cdots & \frac{2rJ_3}{g_n} & \frac{2rJ_2}{g_n} \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

Theorem 2 Let $n \geq 3$. The inverse of $J_n = C_r(J_1, J_2, \cdots, J_n)$ is $J_n^{-1} = C_r(m_1, m_2, \cdots, m_n)$, where

$$m_1 = \frac{2(r-1)rJ_n J_{n-1} + 2rJ_{n-1} - (1-2rJ_{n-1})g_n + J_n - 1}{2r^2 J_n^2 g_n},$$

$$m_2 = \frac{(1-r)J_n + g_n - 1}{J_n g_n},$$

$$m_3 = \frac{1}{g_n} \left((1-J_n - g_n) \frac{(1-rJ_{n+1})^{n-3}}{(2rJ_n)^{n-2}} + \right.$$

$$\left. 2r \sum_{k=1}^{n-3} J_{k+1} \frac{(1-rJ_{n+1})^{k-1}}{(2rJ_n)^k} \right),$$

$$m_4 = \frac{1}{g_n} \left(\frac{((1-J_n - g_n)(rJ_{n+2} - 1) - 4r^2 J_n J_{n-2})}{(2rJ_n)^{n-2}} \right.$$

$$\left. \frac{(1-rJ_{n+1})^{n-4}}{(2rJ_n)^{n-2}} + 4r \sum_{k=1}^{n-4} J_k \frac{(1-rJ_{n+1})^{k-1}}{(2rJ_n)^k} \right),$$

$$m_i = \frac{1}{g_n} \frac{(1-rJ_{n+1})^{n-i}}{(2rJ_n)^{n-i+1}}.$$

$$\begin{cases} \frac{(1-J_n - g_n)(1+r)(r2^{n+2} - 2)}{(2rJ_n)^2} - \\ 2r \left(J_{n-1} + \frac{J_{n-2}(1-rJ_{n+1})}{rJ_n} \right), n \text{ is odd,} \\ \frac{(1-J_n - g_n)(1-r)(r2^n - 1)}{(2rJ_n)^2} - \\ 2r \left(J_{n-1} + \frac{J_{n-2}(1-rJ_{n+1})}{rJ_n} \right), n \text{ is even,} \end{cases}$$

$$i = 5, 6, \cdots, n,$$

and

$$g_n = 1 - rJ_{n+1} + 2r \sum_{k=1}^{n-1} J_k \left(\frac{2rJ_n}{1-rJ_{n+1}} \right)^{n-k-1}.$$

Proof Let

and $G = \text{diag}(1, g_n)$ is a diagonal matrix, where

$$g'_n = \sum_{k=1}^{n-1} J_{k+1} \left(\frac{2rJ_n}{1-rJ_{n+1}} \right)^{n-k-1},$$

and

$$g_n = 1 - rJ_{n+1} + 2r \sum_{k=1}^{n-1} J_k \left(\frac{2rJ_n}{1 - rJ_{n+1}} \right)^{n-k-1}.$$

Then we have

$$P_n \mathbb{J}_n Q_1 Q_2 = G \oplus A,$$

where A is given in Lemma 1. Let $Q_n = Q_1 Q_2$, so that

$$\mathbb{J}_n^{-1} = Q_n (G^{-1} \oplus A^{-1}) P_n.$$

Since the last row elements of the matrix Q_n are

$$\left(0, 1, \frac{1 - J_n}{g_n}, \frac{2rJ_{n-2}}{g_n}, \frac{2rJ_{n-3}}{g_n}, \dots, \frac{2rJ_3}{g_n}, \frac{2rJ_2}{g_n} \right),$$

and $G^{-1} = \text{diag}(1, g_n^{-1})$, using Lemma 1 the last row elements of $\mathbb{J}_n^{-1} = C_r(m_1, m_2, \dots, m_n)$ are given by the following equations:

$$m_2 = ((1-r)J_n + g_n - 1) / J_n g_n,$$

$$m_3 = \frac{1}{g_n} \left((1 - J_n - g_n) \frac{(1 - rJ_{n+1})^{n-3}}{(2rJ_n)^{n-2}} + 2r \sum_{k=1}^{n-3} J_{k+1} \frac{(1 - rJ_{n+1})^{k-1}}{(2rJ_n)^k} \right).$$

$$m_4 = \frac{1}{g_n} \left(\frac{((1 - J_n - g_n)(rJ_{n+2} - 1) - 4r^2 J_n J_{n-2})}{(2rJ_n)^{n-2}} \cdot \frac{(1 - rJ_{n+1})^{n-4}}{(2rJ_n)^{n-2}} + 4r \sum_{k=1}^{n-4} J_k \frac{(1 - rJ_{n+1})^{k-1}}{(2rJ_n)^k} \right),$$

$$m_5 = (1/g_n)(1 - rJ_{n+1})^{n-5} / (2rJ_n)^{n-4}.$$

$$\left\{ \begin{array}{l} \frac{(1 - J_n - g_n)(1+r)(r \cdot 2^{n+2} - 2)}{(2rJ_n)^2} - \\ 2r \left(J_{n-1} + \frac{J_{n-2}(1 - rJ_{n+1})}{rJ_n} \right), \quad n \text{ is odd,} \\ \frac{(1 - J_n - g_n)(1-r)(r \cdot 2^n - 1)}{(2rJ_n)^2} - \\ 2r \left(J_{n-1} + \frac{J_{n-2}(1 - rJ_{n+1})}{rJ_n} \right), \quad n \text{ is even,} \end{array} \right.$$

$$m_i = \frac{1}{g_n} \frac{(1 - rJ_{n+1})^{n-i}}{(2rJ_n)^{n-i+1}}.$$

$$\left\{ \begin{array}{l} \frac{(1 - J_n - g_n)(1+r)(r \cdot 2^{n+2} - 2)}{(2rJ_n)^2} - \\ 2r \left(J_{n-1} + \frac{J_{n-2}(1 - rJ_{n+1})}{rJ_n} \right), \quad i \text{ is odd,} \\ \frac{(1 - J_n - g_n)(1-r)(r \cdot 2^n - 1)}{(2rJ_n)^2} - \\ 2r \left(J_{n-1} + \frac{J_{n-2}(1 - rJ_{n+1})}{rJ_n} \right), \quad i \text{ is even,} \end{array} \right.$$

$$m_n = \frac{1}{g_n} \frac{1}{(2rJ_n)}.$$

$$\left\{ \begin{array}{l} \frac{(1 - J_n - g_n)(1+r)(r \cdot 2^{n+2} - 2)}{(2rJ_n)^2} - \\ 2r \left(J_{n-1} + \frac{J_{n-2}(1 - rJ_{n+1})}{rJ_n} \right), \quad n \text{ is odd,} \\ \frac{(1 - J_n - g_n)(1-r)(r \cdot 2^n - 1)}{(2rJ_n)^2} - \\ 2r \left(J_{n-1} + \frac{J_{n-2}(1 - rJ_{n+1})}{rJ_n} \right), \quad n \text{ is even,} \end{array} \right.$$

$$m_1 = \frac{2(r-1)rJ_n J_{n-1} + 2rJ_{n-1} - (1 - 2rJ_{n-1})g_n + J_n - 1}{2r^2 J_n^2 g_n}.$$

Hence, the proof is completed.

2 Determinant and inverse of r -circulant matrix with the Jacobsthal-Lucas numbers

In this section, we give a determinant formula for the r -circulant matrix $j_n = C_r(j_0, j_1, \dots, j_{n-1})$ firstly. Then, we use the fact that the inverse of a nonsingular r -circulant matrix is also r -circulant matrix, and obtained the formula for the inverse of the r -circulant matrix j_n .

Theorem 3 Let $n \geq 3$. Then we have

$$\det j_n = (2 - rj_n)^{n-2} (4 - rj_{n-1}) + r \sum_{k=1}^{n-2} [(2j_{k+1} - j_k)(1 + 2rj_{n-1})^{n-k-1} (2 - rj_n)^{k-1}], \quad (2)$$

where j_n is the n th Jacobsthal-Lucas number.

So j_n is nonsingular for all $n \geq 1$.

Proof By direct computation $\det j_3$ satisfied (2).

In the case $n > 3$, let

$$K_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1/2r & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -2r & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & -2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 1 & -1 & \ddots & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \left(\frac{1+2rj_{n-1}}{2-rj_n}\right)^{n-2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \left(\frac{1+2rj_{n-1}}{2-rj_n}\right)^{n-3} & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & \left(\frac{1+2rj_{n-1}}{2-rj_n}\right)^{n-4} & 0 & 0 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \left(\frac{1+2rj_{n-1}}{2-rj_n}\right)^2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \left(\frac{1+2rj_{n-1}}{2-rj_n}\right) & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then we have the following relation:

$$U_n = K_n j_n M_1 = \begin{bmatrix} j_1 & y'_n & -j_{n-1} & -j_{n-2} & \cdots & -j_4 & -j_3 & -j_2 \\ y_n & 1/2rj_{n-1} - j_0 & r(1/2j_{n-2} - j_{n-1}) & \cdots & r(1/2j_4 - j_5) & r(1/2j_3 - j_4) & r(1/2j_2 - j_3) \\ & 1+2rj_{n-1} & rj_n - 2 & & & & \\ & & 1+2rj_{n-1} & \ddots & & & 0 \\ & & & \ddots & \ddots & & \\ & & & & \ddots & rj_n - 2 & \\ & & & & & 1+2rj_{n-1} & rj_n - 2 \\ & & & & & & 1+2rj_{n-1} \end{bmatrix},$$

which is a upper triangular, where

$$y'_n = \sum_{k=1}^{n-1} j_k \left(\frac{1+2rj_{n-1}}{2-rj_n} \right)^{n-k-1},$$

and

$$y_n = \frac{1}{2}(4-rj_{n-1}) + \frac{1}{2}r \sum_{k=1}^{n-2} (2j_{k+1} - j_k) \left(\frac{1+2rj_{n-1}}{2-rj_n} \right)^{n-k-1}.$$

Then we obtained

$$\det U_n = \det K_n \det j_n \det M_1 = 2y_n (1+2rj_{n-1})^{n-2}.$$

$$\text{While } \det K_n = (-1)^{\frac{(n-1)(n-2)}{2}},$$

$$\text{and } \det M_1 = \left(\frac{1+2rj_{n-1}}{2-rj_n} \right)^{n-2} (-1)^{\frac{(n-1)(n-2)}{2}}.$$

$$\text{We have } \det K_n \det M_1 = \left(\frac{1+2rj_{n-1}}{2-rj_n} \right)^{n-2}. \text{ Thus}$$

$$\det j_n = (2-rj_n)^{n-2} (4-rj_{n-1}) + r \sum_{k=1}^{n-2} [(2j_{k+1} - j_k) \cdot (1+2rj_{n-1})^{n-k-1} (2-rj_n)^{k-1}].$$

Lemma 2 Let the matrix $B = [b_{ij}]_{i,j=1}^{n-2}$ is defined as

$$b_{ij} = \begin{cases} 1+2rj_{n-1}, & i=j, \\ rj_n-2, & j=i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then the inverse $B^{-1} = [b'_{ij}]_{i,j=1}^{n-2}$ is equal to

$$b'_{ij} = \begin{cases} \frac{(2-rj_n)^{j-i}}{(1+2rj_{n-1})^{j-i+1}}, & j \geq i, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4 Let $n \geq 3$. The inverse of $j_n = C_r(j_0, j_1, \dots, j_{n-1})$ is $j_n^{-1} = C_r(h_0, h_1, \dots, h_{n-1})$, where

$$h_0 = \frac{1}{2y_n} \left(\frac{9rj_n + (10-8rj_{n-2})y_n - 18}{(1+2rj_{n-1})^2} \right),$$

$$h_1 = \frac{r}{2y_n} \left(\frac{4y_n - 9}{1+2rj_{n-1}} \right),$$

$$h_2 = \frac{1}{2y_n} \left((4-rj_{n-1}-2y_n) \frac{(2-rj_n)^{n-3}}{(1+2rj_{n-1})^{n-2}} + r \sum_{k=1}^{n-3} (2j_{k+2} - j_{k+1}) \frac{(2-rj_n)^{k-1}}{(1+2rj_{n-1})^k} \right),$$

$$h_3 = (1/2y_n)[((4-j_{n-1}-2y_n)(rj_{n+1}-1) - r(2j_{n-1}-j_{n-2})(1+2rj_{n-1})) \frac{(2-rj_n)^{n-4}}{(1+2rj_{n-1})^{n-2}} + 2r \sum_{k=1}^{n-4} (2j_{k+1} - j_k) \frac{(2-rj_n)^{k-1}}{(1+2rj_{n-1})^k}],$$

$$h_i = (2-rj_n)^{n-i-1} / (2y_n (1+2rj_{n-1})^{n-i}).$$

$$\begin{cases} 9(r+1)(r \cdot 2^n - 1)(4-rj_{n-1}-2y_n) - r(2j_n - j_{n-i}) - \frac{2r(2j_{n-1} - j_{n-2})(2-rj_n)}{1+2rj_{n-1}}, & n \text{ is odd,} \\ 9(r-1)(1-r \cdot 2^n)(4-rj_{n-1}-2y_n) - \frac{2r(2j_{n-1} - j_{n-2})(2-rj_n)}{1+2rj_{n-1}}, & n \text{ is even.} \end{cases}$$

$$i = 4, 5, 6, \dots, n-1.$$

and

$y_n = \frac{1}{2}(4 - rj_{n-1}) + \frac{1}{2}r \sum_{k=1}^{n-2} (2j_{k+1} - j_k) \left(\frac{1+2rj_{n-1}}{2-rj_n} \right)^{n-k-1}$. elements of $j_n^{-1} = C_r(h_0, h_1, \dots, h_{n-1})$ is $(h_1, h_2, \dots, h_{n-1}, h_0)$. The proof is completed.

Proof Let

$$M_2 = \begin{bmatrix} 1 & -1/2y'_n & \frac{y'_n}{4y_n}(rj_{n-1}-4)+1/2j_{n-1} & r(j_{n-2}-2j_{n-1})\frac{y'_n}{4y_n}+1/2j_{n-2} & \cdots & r(j_2-j_3)\frac{y'_n}{4y_n}+1/2j_2 \\ 0 & 1 & \frac{4-rj_{n-1}}{2y_n} & \frac{r(2j_{n-1}-j_{n-2})}{2y_n} & \cdots & \frac{r(2j_3-j_2)}{2y_n} \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and $Y = \text{diag}(2, y_n)$ is a diagonal matrix, where

$$y'_n = \sum_{k=1}^{n-1} j_k \left(\frac{1+2rj_{n-1}}{2rj_n} \right)^{n-k-1},$$

and

$$y_n = \frac{1}{2}(4 - rj_{n-1}) + \frac{1}{2}r \sum_{k=1}^{n-2} (2j_{k+1} - j_k) \left(\frac{1+2rj_{n-1}}{2-rj_n} \right)^{n-k-1},$$

then we have

$$K_n j_n M_1 M_2 = Y \oplus B,$$

where B is given in Lemma 2. Let $M_n = M_1 M_2$, so that

$$j_n^{-1} = M_n(Y^{-1} \oplus B^{-1})K_n.$$

Since the last row elements of the matrix M_n are

$$\left(0, 1, \frac{4-rj_{n-1}}{2y_n}, \frac{r(2j_{n-1}-j_{n-2})}{2y_n}, \frac{r(2j_{n-2}-j_{n-3})}{2y_n}, \dots, \frac{r(2j_4-j_3)}{2y_n}, \frac{r(2j_3-j_2)}{2y_n} \right),$$

and $Y^{-1} = \text{diag}(1/2, y_n^{-1})$, using Lemma 2 the last row

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关于 Jacobsthal 数和 Jacobsthal-Lucas 数的 r -循环矩阵的行列式和逆

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摘要: $J_n = C_r(J_1, J_2, \dots, J_n)$ 和 $j_n = C_r(j_0, j_1, \dots, j_{n-1})$ 是 n 阶 r -循环矩阵, 其中 $J_1, J_2, \dots, J_n$ 是 Jacobsthal 数, $j_0, j_1, \dots, j_{n-1}$ 是 Jacobsthal-Lucas 数, 研究得出了 J_n 和 j_n 的关于 Jacobsthal 数和 Jacobsthal-Lucas 数的行列式的值. 同时, 计算出了矩阵 J_n 和 j_n 的逆.

关键词: r -循环矩阵; 行列式; 逆; Jacobsthal 数; Jacobsthal-Lucas 数

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