

二叉树模型下欧式期权价值关于波动率的单调性研究

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摘要: 研究了在市场无套利情况下, 二叉树模型的欧式期权价格与标的资产波动率的单调性问题. 首先给出了2个新的组合公式, 然后借助该组合公式证明了欧式期权价格与标的资产波动率存在单调递增关系.

关键词: 二叉树模型; 期权定价; 波动率

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期权作为一种金融衍生产品, 其定价是金融市场一个重要的研究领域. 近几十年来, 期权定价问题已得到广泛研究. Black 等^[1]提出了 Black-Scholes 期权定价模型, 虽然 Black-Scholes 期权定价模型有很多优点, 但其推导过程复杂. 随后 Cox 等^[2]提出了一种简单的对离散时间期权的定价方法, 被称为 C-R-R 二项式期权定价模型. 近些年很多国内外学者在文献[2]基础上研究了二项式模型的期权定价问题^[3-7]. 众所周知, 欧式期权的价格在 C-R-R 二项式期权定价模型中依赖于标的资产的波动性, 即欧式期权价格与标的资产波动率成单调递增关系. 近年来有关期权定价波动性的研究成果颇丰^[8-10]. 陈怡^[4]证明了在单期二叉树模型下欧式看涨期权关于波动率的单调关系, 但没有涉及多周期模型. Reynaerts 等^[11]通过对二叉树模型的期权定价进行敏感性分析, 得出了多周期时欧式看涨期权的价格不一定是波动率严格增加函数的结论, 但这种方法非常复杂.

本文利用2个新的组合公式, 更简便地证明了多周期二叉树模型下欧式期权价格与标的资产波动率的单调关系.

1 期权定价模型

考虑一个在风险中性测度意义下的没有交易成本和税收, 且不允许有套利机会的金融期权市

场. 把期权的生存区间 $[0, T]$, 细分为 N 个子区间 $0 = t_0 < t_1 < \dots < t_N = T$, 在每个区间 $[t_n, t_{n+1}]$ ($0 \leq n \leq N-1$) 上, 假设原生资产 S 的演化适合单时段双状态模型, 且在期末时刻 S_T 只有2种可能: S_T^u 和 S_T^d . 其中 S_T 表示 T 时刻原生资产的价格, u 表示上升波动因子, d 表示下降波动因子.

对于欧式期权, 在支付红利情况下, 考虑形式为如下的欧式看涨期权定价公式^[12]:

$$V = \frac{1}{\eta^N} \sum_{i=0}^{\hat{\alpha}} C_N^i \hat{q}^{N-i} (1-\hat{q})^i - \frac{K}{\rho^N} \sum_{i=0}^{\hat{\alpha}} C_N^i q^{N-i} (1-q)^i, \quad (1)$$

以及形式如下的欧式看跌期权定价公式^[12]:

$$P = \frac{K}{\rho^N} \sum_{i=\hat{\alpha}+1}^N C_N^i q^{N-i} (1-q)^i - \frac{1}{\eta^N} \sum_{i=\hat{\alpha}+1}^N C_N^i \hat{q}^{N-i} (1-\hat{q})^i, \quad (2)$$

式中: $\hat{\alpha} = \max \{ \alpha | u^{N-\alpha} d^\alpha - K \geq 0, 0 \leq \alpha \leq N \}$, $q = (\rho/\eta - d)/(u-d)$, $\hat{q} = u\eta q/\rho$, 这里 K 表示敲定价格, 且 $\rho = 1 + r\Delta t$, $\eta = 1 + m\Delta t$, r 表示无风险利率, m 表示红利率.

2 二叉树模型下欧式期权定价与波动率关系

给出2个新的组合公式, 这2个组合公式可应用于定理1的证明.

引理 1

$$\frac{d}{dx} \left[\sum_{i=0}^{\beta} C_N^i f^{N-i}(x) (1-f(x))^i \right] = C_N^\beta (N-\beta) f^{N-\beta-1}(x) f'(x) (1-f(x))^\beta, \quad (3)$$

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$$\frac{d}{dx} \left[\sum_{i=\beta+1}^N C_N^i f^{N-i}(x) (1-f(x))^i \right] = -C_N^{N-\beta-1} \cdot (\beta+1)(1-f(x))^\beta f'(x) f^{N-\beta-1}(x). \quad (4)$$

证明

$$\begin{aligned} \frac{d}{dx} \left[\sum_{i=0}^{\beta} C_N^i f^{N-i}(x) (1-f(x))^i \right] &= \sum_{i=0}^{\beta} C_N^i (N-i) f^{N-i-1}(x) f'(x) (1-f(x))^i - \\ &\sum_{i=1}^{\beta} i C_N^i f^{N-i}(x) f'(x) (1-f(x))^{i-1} = \\ &\sum_{i=0}^{\beta-1} f^{N-i-1}(x) f'(x) (1-f(x))^i [C_N^i (N-i) - \\ &C_N^{i+1} (i+1)] + C_N^{\beta} (N-\beta) f^{N-\beta-1}(x) f'(x) \cdot \\ &(1-f(x))^\beta = C_N^{\beta} (N-\beta) \cdot \\ &f^{N-\beta-1}(x) f'(x) (1-f(x))^\beta. \end{aligned}$$

同理可证

$$\begin{aligned} \frac{d}{dx} \left[\sum_{i=\beta+1}^N C_N^i f^{N-i}(x) (1-f(x))^i \right] &= \\ \frac{d}{dx} \left[\sum_{j=0}^{N-\beta-1} C_N^{N-j} f^j(x) (1-f(x))^{N-j} \right] &= \\ -C_N^{N-\beta-1} (\beta+1) (1-f(x))^\beta f'(x) f^{N-\beta-1}(x). \end{aligned}$$

引理 1 不仅可用于欧式期权价格关于波动率的单调性证明, 在其他研究中也有广泛应用.

定理 1 在支付红利情况下, $\hat{\alpha} = \max\{\alpha \mid u^{N-\alpha} \cdot d^\alpha - K \geq 0, 0 \leq \alpha \leq N\}$, 若 $d < 1 < \rho < u$, $d\eta < \rho < u\eta$, $\eta > 1$, 则欧式期权的价格式(1)和式(2)随着波动率的增加而单调增. 特别当 $\hat{\alpha} \neq N$ 且 $K \neq u^{N-\hat{\alpha}} d^{\hat{\alpha}}$ 时, 欧式期权的价格对于波动率严格单调.

证明 若将欧式看涨期权定价式(1)对 u 求偏导, 根据式(3)可得:

$$\frac{\partial V}{\partial u} = C_N^{\hat{\alpha}} (N - \hat{\alpha}) \cdot \left[\frac{1}{\eta^N} \hat{q}^{N-\hat{\alpha}-1} \hat{q}'_d (1-\hat{q})^{\hat{\alpha}} - \frac{K}{\rho^N} q^{N-\hat{\alpha}-1} q'_d (1-q)^{\hat{\alpha}} \right], \quad (5)$$

式中:

$$q'_u = \frac{d\eta - \rho}{\eta(u-d)^2}, \quad \hat{q}'_u = \frac{(d\eta - \rho)d}{\rho(u-d)^2}, \quad (6)$$

这里 q'_u 和 $\hat{q}'_u$ 表示 q 和 $\hat{q}$ 对 u 求导.

将式(6)代入式(5)整理可得:

$$\frac{\partial V}{\partial u} = \frac{C_N^{\hat{\alpha}} (N - \hat{\alpha}) (\rho - d\eta)^{N-\hat{\alpha}} (u\eta - \rho)^{\hat{\alpha}}}{\eta^N \rho^N (u-d)^{N+1}} \cdot$$

$$(K - u^{N-\hat{\alpha}-1} d^{\hat{\alpha}+1}),$$

因为 $N \geq \hat{\alpha}$, 又由于 $d < 1 < \rho < u$, $d\eta < \rho < u\eta$, $\eta > 1$, 因此有:

$$N \geq \hat{\alpha}, \quad \eta^N > 0, \quad \rho^N > 0,$$

$$(\rho - d\eta)^{N-\hat{\alpha}} > 0, \quad (u\eta - \rho)^{\hat{\alpha}} > 0, \quad (u-d)^{N+1} > 0, \quad (7)$$

由于

$$\hat{\alpha} = \max\{\alpha \mid u^{N-\alpha} d^\alpha - K \geq 0, 0 \leq \alpha \leq N\},$$

可得:

$$K - u^{N-\hat{\alpha}-1} d^{\hat{\alpha}+1} > 0. \quad (8)$$

根据式(7)和式(8)可以得到 $\partial V / \partial u \geq 0$. 特别当 $\hat{\alpha} \neq N$ 时, $\partial V / \partial u > 0$.

若将欧式看涨期权定价式(1)对 d 求偏导, 根据式(3)可得:

$$\frac{\partial V}{\partial d} = C_N^{\hat{\alpha}} (N - \hat{\alpha}) \cdot$$

$$\left[\frac{1}{\eta^N} \hat{q}^{N-\hat{\alpha}-1} \hat{q}'_d (1-\hat{q})^{\hat{\alpha}} - \frac{K}{\rho^N} q^{N-\hat{\alpha}-1} q'_d (1-q)^{\hat{\alpha}} \right], \quad (9)$$

式中:

$$q'_d = \frac{\rho - \eta u}{\eta(u-d)^2}, \quad \hat{q}'_d = \frac{u(\rho - u\eta)}{\rho(u-d)^2}, \quad (10)$$

这里 q'_d 和 $\hat{q}'_d$ 表示 q 和 $\hat{q}$ 对 d 求导.

将式(10)代入式(9)整理可得:

$$\frac{\partial V}{\partial d} = \frac{C_N^{\hat{\alpha}} (N - \hat{\alpha}) (\rho - d\eta)^{N-\hat{\alpha}-1} (u\eta - \rho)^{\hat{\alpha}+1}}{\eta^N \rho^N (u-d)^{N+1}} \cdot$$

$$(K - u^{N-\hat{\alpha}} d^{\hat{\alpha}}),$$

因为 $N \geq \hat{\alpha}$, 又因 $d < 1 < \rho < u$, $d\eta < \rho < u\eta$, $\eta > 1$, 可得:

$$\frac{C_N^{\hat{\alpha}} (N - \hat{\alpha}) (\rho - d\eta)^{N-\hat{\alpha}-1} (u\eta - \rho)^{\hat{\alpha}+1}}{\eta^N \rho^N (u-d)^{N+1}} \geq 0, \quad (11)$$

由于

$$\hat{\alpha} = \max\{\alpha \mid u^{N-\alpha} d^\alpha - K \geq 0, 0 \leq \alpha \leq N\},$$

可得:

$$K - u^{N-\hat{\alpha}} d^{\hat{\alpha}} \leq 0, \quad (12)$$

那么由式(11)和式(12)可得:

$$\frac{\partial V}{\partial d} \leq 0,$$

特别当 $\hat{\alpha} \neq N$ 且 $K \neq u^{N-\hat{\alpha}} d^{\hat{\alpha}}$ 时, $\partial V / \partial d < 0$.

若将欧式看跌期权定价式(2)对 u 和 d 分别求偏导, 根据式(4)类似可得:

$$\frac{\partial P}{\partial u} = \frac{C_N^{N-\hat{\alpha}-1}(\hat{\alpha}+1)(\rho-d\eta)^{N-\hat{\alpha}}(u\eta-\rho)^{\hat{\alpha}}}{\eta^N \rho^N (u-d)^{N+1}}.$$

$$(K-u^{N-\hat{\alpha}-1}d^{\hat{\alpha}+1}) > 0,$$

$$\frac{\partial P}{\partial d} = \frac{C_N^{N-\hat{\alpha}-1}(\hat{\alpha}+1)(\rho-d\eta)^{N-\hat{\alpha}-1}(u\eta-\rho)^{\hat{\alpha}+1}}{\eta^N \rho^N (u-d)^{N+1}}.$$

$$(K-u^{N-\hat{\alpha}}d^{\hat{\alpha}}) \leq 0,$$

特别当 $K \neq u^{N-\hat{\alpha}}d^{\hat{\alpha}}$ 时, $\partial P / \partial d < 0$.

进一步考虑特殊情况, 即 $u=1/d$ 时, 将看涨期权定价式(1)对 u 求偏导, 根据式(3)可得:

$$\begin{aligned} \frac{\partial V}{\partial u} = & C_N^{\hat{\alpha}}(N-\hat{\alpha}) \left[\frac{1}{\eta^N} \hat{q}^{N-\hat{\alpha}-1} \hat{q}'_u (1-\hat{q})^{\hat{\alpha}} - \right. \\ & \left. \frac{K}{\rho^N} q^{N-\hat{\alpha}-1} q'_u (1-q)^{\hat{\alpha}} \right], \end{aligned} \quad (13)$$

式中:

$$\begin{aligned} q = & \frac{\rho u - \eta}{\eta(u^2 - 1)}, \quad \hat{q} = \frac{u(\rho u - \eta)}{\rho(u^2 - 1)}, \\ q'_u = & \frac{2u\eta - \rho - \rho u^2}{\eta(u^2 - 1)^2}, \quad \hat{q}'_u = \frac{\eta u^2 - 2\rho u + \eta}{\rho(u^2 - 1)^2}, \end{aligned} \quad (14)$$

这里 q'_u 和 $\hat{q}'_u$ 表示 q 和 $\hat{q}$ 对 u 求导.

将式(14)代入式(13)整理可得:

$$\begin{aligned} \frac{\partial V}{\partial u} = & \frac{C_N^{\hat{\alpha}}(N-\hat{\alpha})(u\eta-\rho)^{\hat{\alpha}}(\rho u-\eta)^{N-\hat{\alpha}-1}}{\eta^N \rho^N (u^2-1)^{N+1}} \cdot \\ & [u^{N-\hat{\alpha}-1}(\eta u^2-2\rho u+\eta)-Ku^{\hat{\alpha}}(2u\eta-\rho-\rho u^2)]. \end{aligned}$$

因为

$$\hat{\alpha} \leq N, d\eta < \rho < u\eta, d < 1 < \rho < u, \eta > 1,$$

所以

$$\frac{C_N^{\hat{\alpha}}(N-\hat{\alpha})(u\eta-\rho)^{\hat{\alpha}}(\rho u-\eta)^{N-\hat{\alpha}-1}}{\eta^N \rho^N (u^2-1)^{N+1}} \geq 0, \quad (15)$$

由 $\hat{\alpha}$ 定义可知:

$$u^{N-2\hat{\alpha}-2} < K \leq u^{N-2\hat{\alpha}}, \quad (16)$$

当 $K = u^{N-2\hat{\alpha}}$ 时,

$$\begin{aligned} [u^{N-\hat{\alpha}-1}(\eta u^2-2\rho u+\eta)-Ku^{\hat{\alpha}}(2u\eta-\rho-\rho u^2)] = \\ u^{N-\hat{\alpha}-1}(\rho u-\eta)(u^2-1) > 0, \end{aligned} \quad (17)$$

当 $K = u^{N-2\hat{\alpha}-2}$ 时, 同理可证

$$\begin{aligned} u^{N-\hat{\alpha}-1}(\eta u^2-2\rho u+\eta)- \\ Ku^{\hat{\alpha}}(2u\eta-\rho-\rho u^2) > 0, \end{aligned} \quad (18)$$

又由于 $\partial V / \partial u$ 是关于 K 的线性函数, 所以由式

(15)~(18)可得 $K \in (u^{N-2\hat{\alpha}-2}, u^{N-2\hat{\alpha}}]$ 时, 有 $\partial V / \partial u \geq 0$.

特别 $\hat{\alpha} \neq N$ 时, $\partial V / \partial u > 0$.

再将欧式看涨期权定价式(1)对 d 求偏导, 根据式(3)可得:

$$\begin{aligned} \frac{\partial V}{\partial d} = & C_N^{\hat{\alpha}}(N-\hat{\alpha}) \left[\frac{1}{\eta^N} \hat{q}^{N-\hat{\alpha}-1} \hat{q}'_d (1-\hat{q})^{\hat{\alpha}} - \right. \\ & \left. \frac{K}{\rho^N} q^{N-\hat{\alpha}-1} q'_d (1-q)^{\hat{\alpha}} \right], \end{aligned} \quad (19)$$

式中:

$$\begin{aligned} q = & \frac{(\rho-d\eta)d}{\eta(1-d^2)}, \quad \hat{q} = \frac{\rho-d\eta}{\rho(1-d^2)}, \\ q'_d = & \frac{\rho-2\eta d + \rho d^2}{\eta(1-d^2)^2}, \quad \hat{q}'_d = \frac{2d\rho-\eta-\eta d^2}{\rho(1-d^2)^2}, \end{aligned} \quad (20)$$

将式(20)代入式(19), 类似可得 $K \in (u^{N-2\hat{\alpha}-2}, u^{N-2\hat{\alpha}}]$,

特别当 $\hat{\alpha} \neq N$ 时, 有:

$$\begin{aligned} \frac{\partial V}{\partial d} = & C_N^{\hat{\alpha}}(N-\hat{\alpha}) \frac{(\rho-\eta d)^{N-\hat{\alpha}-1}(\eta-pd)^{\hat{\alpha}}}{\eta^N \rho^N (1-d^2)^{N+1}} \cdot \\ & [d^{\hat{\alpha}}(2d\rho-\eta-\eta d^2)- \\ & Kd^{N-\hat{\alpha}-1}(\rho-2\eta d + \rho d^2)] = \\ & -u^2 \frac{C_N^{\hat{\alpha}}(N-\hat{\alpha})(u\eta-\rho)^{\hat{\alpha}}(\rho u-\eta)^{N-\hat{\alpha}-1}}{\eta^N \rho^N (u^2-1)^{N+1}} \cdot \\ & [u^{N-\hat{\alpha}-1}(\eta u^2-2\rho u+\eta)- \\ & Ku^{\hat{\alpha}}(2u\eta-\rho-\rho u^2)] < 0. \end{aligned}$$

若欧式看跌期权定价式(2)对 u 和 d 求偏导, 根据式(4)类似可得 $K \in (u^{N-2\hat{\alpha}-2}, u^{N-2\hat{\alpha}}]$ 时,

$$\begin{aligned} \frac{\partial P}{\partial u} = & \frac{C_N^{N-\hat{\alpha}-1}(\hat{\alpha}+1)(u\eta-\rho)^{\hat{\alpha}}(\rho u-\eta)^{N-\hat{\alpha}-1}}{\eta^N \rho^N (u^2-1)^{N+1}} \cdot \\ & [u^{N-\hat{\alpha}-1}(\eta u^2-2\rho u+\eta)- \\ & Ku^{\hat{\alpha}}(2u\eta-\rho-\rho u^2)] > 0, \\ \frac{\partial P}{\partial d} = & C_N^{N-\hat{\alpha}-1}(\hat{\alpha}+1) \frac{(\rho-\eta d)^{N-\hat{\alpha}-1}(\eta-pd)^{\hat{\alpha}}}{\eta^N \rho^N (1-d^2)^{N+1}} \cdot \\ & [d^{\hat{\alpha}}(2d\rho-\eta-\eta d^2)- \\ & Kd^{N-\hat{\alpha}-1}(\rho-2\eta d + \rho d^2)] = \\ & -u^2 \frac{C_N^{N-\hat{\alpha}-1}(\hat{\alpha}+1)(u\eta-\rho)^{\hat{\alpha}}(\rho u-\eta)^{N-\hat{\alpha}-1}}{\eta^N \rho^N (u^2-1)^{N+1}} \cdot \\ & [u^{N-\hat{\alpha}-1}(\eta u^2-2\rho u+\eta)- \end{aligned}$$

$$Ku^{\hat{\alpha}}(2u\eta - \rho - \rho u^2)] < 0.$$

从证明可知, 欧式期权价格对波动因子 u 求导大于等于零, 对波动因子 d 求导小于等于零, 而波动率与 u 呈正相关, 与 d 呈负相关, 即证得欧式期权价格随着波动率的增加而单调增. 特别是 $\hat{\alpha} \neq N$ 且 $K \neq u^{N-\hat{\alpha}}d^{\hat{\alpha}}$ 时, 欧式期权价格对波动因子 u 求导严格大于零, 且对波动因子 d 求导严格小于零, 即证得此条件下欧式期权价格对波动率严格单调.

不支付红利的欧式期权价格是支付红利的欧式期权价格 $\eta = 1$ 时的情况, 同理可证不支付红利的欧式期权价格也有相同结论.

3 结语

本文研究了在金融市场无套利情况下二叉树模型的欧式期权价格与波动率的单调性问题. 研究结果表明, 欧式期权价格与波动率存在单调递增关系, 特别当 $\hat{\alpha} \neq N$ 且 $K \neq u^{N-\hat{\alpha}}d^{\hat{\alpha}}$ 时, 欧式期权价格对波动率严格单调.

参考文献:

- [1] Black F, Scholes M. The pricing of options and corporate liabilities[J]. Journal of Political Economy, 1973, 81(3): 637-654.
- [2] Cox J C, Ross S A, Rubinstein M. Option pricing: A simplified approach[J]. Journal of Financial Economics, 1979, 7(3):229-263.
- [3] Muzzioli S, Torricelli C. A multiperiod binomial model for pricing options in a vague world[J]. Journal of Economic Dynamics and Control, 2004, 28(5):861-887.
- [4] 陈怡. 关于期权定价的模糊二叉树模型及其应用[D]. 天津: 天津大学, 2007.
- [5] 陈怡. 关于欧式看涨期权的模糊二叉树模型[J]. 哈尔滨商业大学学报(社会科学版), 2007(6):10-12.
- [6] 孔凡秋. 期权的二项式定价模型研究[J]. 经济研究导刊, 2014(4):134-135; 155.
- [7] 冯晶晶, 樊亚云, 邢瑞芳. 二叉树从二期模型到 n 期模型的扩展[J]. 重庆理工大学学报(自然科学), 2016, 30(10):181-184.
- [8] 国泰君安证券. 五大因素影响期权价格[N]. 证券时报, 2014-07-24(A15).
- [9] 陈雨童. 基于波动率预测的期权定价研究[D]. 上海: 复旦大学, 2013.
- [10] 龚谊洲. 基于可变波动率的期权定价研究[D]. 武汉: 华中师范大学, 2019.
- [11] Reynaerts H, Vanmaele M. A sensitivity analysis for the pricing of European call options in a binary tree model [C]//Proceedings of the Third International Symposium on Imprecise Probabilities and Their Applications, 2003:14-17.
- [12] 姜礼尚. 期权定价的数学模型和方法[M]. 2 版. 北京: 高等教育出版社, 2008:24-39.

Monotonicity of volatility in European option pricing under binomial tree model

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Abstract: In this paper, we study the monotonicity of European option price and underlying asset volatility of the binary tree model for the case of no arbitrage. Firstly, we derive two new combination formulas. Then, based on these two combination formulas, the monotonic incrementing relation between European option pricing and underlying asset volatility is proved.

Key words: binary tree model; option pricing; volatility

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