

具有部分分数阶扩散项的三维 Navier-Stokes 方程的适定性

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摘要: 众所周知, 具有超耗散 $(-\Delta)^{5/4}$ 的 Navier-Stokes 方程是适定的. 许多学者研究了具有弱超耗散的问题. 去除 Navier-Stokes 方程中一些不同方向的超耗散分量, 在一些额外的条件下, 可以证明解的存在性和唯一性. 本文推导出从 u_2 和 u_3 的方程中去除沿 x_3 方向的超耗散问题解的存在性和唯一性. 需要指出的是, 如果从所有方程中去除沿 x_3 方向的超耗散, 则本方法无法获得此问题的适定性结果.

关键词: Navier-Stokes 方程; 部分耗散; 分数阶耗散; 超耗散; 适定性

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Navier-Stokes 方程描述了黏性流体的运动, 而三维不可压方程解的适定性问题仍然是公开的难题. 1934 年, Leray^[1]证明了不可压 Navier-Stokes 方程全局弱解的存在性. 到目前为止, 最优的正则性结果仍然归功于 Caffarelli 等^[2], 他们证明了 Navier-Stokes 方程的任何适当弱解的奇异集具有一维 Hausdorff 零测度. 后来, Lin^[3]使用紧性方法证明了相同的结果. Vasseur^[4]使用 De Giorgi 为椭圆方程引入的方法证明了同样的结果. 由于没有足够的耗散来控制方程的非线性性, 因此标准的 Navier-Stokes 方程是超临界的.

为了增强 Navier-Stokes 方程的耗散, 诸多学者进行了一系列改进. 考虑具有超耗散的广义 Navier-Stokes 方程

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - \nu(-\Delta)^\alpha \mathbf{u}, \\ x \in \mathbb{R}^3, t > 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \end{cases} \quad (1)$$

其中 $\mathbf{u} = (u_1, u_2, u_3)$ 表示流体的速度场, p 代表压力. 如果 $\alpha \geq 5/4$ 且初始数据是光滑的, 则该方程具有全局光滑解并且解是具有唯一性的^[5-8]. 2009 年, Tao^[9]研究了具有以下对数形式的超临界超耗

散项的 Navier-Stokes 方程:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - \nu \frac{(-\Delta)^{5/4}}{\log^{1/2}(2-\Delta)} \mathbf{u}, \\ x \in \mathbb{R}^3, t > 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x). \end{cases} \quad (2)$$

Tao 改进了上述结果并发现方程(2)对任何光滑且有紧支集的初始数据都有全局光滑解. Barbatto 等^[10]通过对数弱耗散项上引入较弱的条件, 进一步改进了 Tao 的研究结果. 学者们还研究了具有超耗散的不可压磁流体动力学方程, 但结果并不完全一样^[7,11].

一些数学家通过研究具有部分超耗散的 Navier-Stokes 方程来改进方程(1)的结果. Yang 等^[12]减少了方程(1)中的超耗散, 研究了以下具有部分耗散的三维 Navier-Stokes 方程的全局正则性:

$$\begin{cases} \partial_t u_1 + \mathbf{u} \cdot \nabla u_1 = -\partial_1 p - \nu \Lambda_1^{5/2} u_1 - \nu \Lambda_2^{5/2} u_1, \\ \partial_t u_2 + \mathbf{u} \cdot \nabla u_2 = -\partial_2 p - \nu \Lambda_2^{5/2} u_2 - \nu \Lambda_3^{5/2} u_2, \\ \partial_t u_3 + \mathbf{u} \cdot \nabla u_3 = -\partial_3 p - \nu \Lambda_1^{5/2} u_3 - \nu \Lambda_3^{5/2} u_3, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \end{cases} \quad (3)$$

并证明了当 $\mathbf{u}_0 \in H^1(\mathbb{R}^3)$ 时, 方程(3)的强解具有全

局存在性和唯一性. 2021 年, Li 等^[13]证明了 $\mathbf{u}_0 \in H^s(\mathbb{R}^3)$ ($s > 5/2$) 时方程(3) H^s 解的全局存在性和唯一性.

当所有方程中都去除沿 x_3 方向的超耗散时, 可得

$$\begin{cases} \partial_t u_1 + \mathbf{u} \cdot \nabla u_1 = -\partial_1 p - \nu \Lambda_1^{5/2} u_1 - \nu \Lambda_2^{5/2} u_1, \\ \partial_t u_2 + \mathbf{u} \cdot \nabla u_2 = -\partial_2 p - \nu \Lambda_1^{5/2} u_2 - \nu \Lambda_2^{5/2} u_2, \\ \partial_t u_3 + \mathbf{u} \cdot \nabla u_3 = -\partial_3 p - \nu \Lambda_1^{5/2} u_3 - \nu \Lambda_2^{5/2} u_3, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x). \end{cases} \quad (4)$$

方程(4)的适定性仍然是公开的, 其主要困难在于控制 $\|\partial_i u_j \partial_j u_k \partial_i u_k\|_{L^1}$, $i=3, k \neq 3$.

受 Yang 等^[12]研究的启发, 本文从 u_2 和 u_3 的方程中去除沿 x_3 方向的超耗散, 考虑以下不可压 Navier-Stokes 方程的适定性:

$$\begin{cases} \partial_t u_1 + \mathbf{u} \cdot \nabla u_1 = -\partial_1 p - \nu \Lambda_1^{5/2} u_1, \\ \partial_t u_2 + \mathbf{u} \cdot \nabla u_2 = -\partial_2 p - \nu \Lambda_1^{5/2} u_2 - \nu \Lambda_2^{5/2} u_2, \\ \partial_t u_3 + \mathbf{u} \cdot \nabla u_3 = -\partial_3 p - \nu \Lambda_1^{5/2} u_3 - \nu \Lambda_2^{5/2} u_3, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x). \end{cases} \quad (5)$$

本文证明了方程(5)的 H^1 解的全局存在性和唯一性. 需要注意的是, 如果 u_1, u_2 没有沿 x_3 方向的耗散, 即将 $\Lambda_3^{5/2}$ 从 u_1, u_2 的方程中去掉, 则使用此方法不能证明以下问题的适定性:

$$\begin{cases} \partial_t u_1 + \mathbf{u} \cdot \nabla u_1 = -\partial_1 p - \nu \Lambda_1^{5/2} u_1 - \nu \Lambda_2^{5/2} u_1, \\ \partial_t u_2 + \mathbf{u} \cdot \nabla u_2 = -\partial_2 p - \nu \Lambda_1^{5/2} u_2 - \nu \Lambda_2^{5/2} u_2, \\ \partial_t u_3 + \mathbf{u} \cdot \nabla u_3 = -\partial_3 p - \nu \Lambda_1^{5/2} u_3, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x). \end{cases}$$

1 主要结果

首先介绍一些符号. 在本文其余部分, 分数阶拉普拉斯算子 $(-\Delta)^\alpha$ 是通过傅里叶变换定义的^[14]:

$$\widehat{(-\Delta)^\alpha f(\xi)} = |\xi|^{2\alpha} \hat{f}(\xi).$$

另外, Λ_k^γ , $\gamma > 0$ 且 $k=1, 2, 3$, 表示分数阶方向微分算子:

$$\widehat{\Lambda_k^\gamma f(\xi)} = |\xi_k|^\gamma \hat{f}(\xi).$$

用 $\Lambda = (-\Delta)^{1/2}$ 表示 Zygmund 算子, $\|f\|_{L^p}$ 表示 $\|f\|_{L^p(\mathbb{R}^3)}$, $\|f\|_{L_{x_i}^p}$ 表示 x_i 方向的一维 L^p -范数, $\|f\|_{L_{x_i x_j}^p}$ ($i \neq j$) 表示在 $x_i x_j$ 平面上的二维 L^p -范数.

类似地,

$$\|f\|_{L_{x_i}^r L_{x_j}^q L_{x_k}^p} := \left(\int \left(\int \left(\int f^p dx_k \right)^{q/p} dx_j \right)^{r/q} dx_i \right)^{1/r} = \| \| \| f \|_{L_{x_k}^p L_{x_j}^q L_{x_i}^r}.$$

本文的主要结果可表示为如下两个定理.

定理 1 假设 $\mathbf{u}_0 \in H^1(\mathbb{R}^3)$, 则方程(5)存在一个全局解 $\mathbf{u}$, 且满足

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, \infty; H^1), \\ \Lambda_1^{5/4} u_1, \nabla \Lambda_1^{5/4} u_1 &\in L^2(0, \infty; L^2(\mathbb{R}^3)), \\ \Lambda_1^{5/4} u_2, \Lambda_2^{5/4} u_2, \nabla \Lambda_1^{5/4} u_2, \nabla \Lambda_2^{5/4} u_2 &\in L^2(0, \infty; L^2(\mathbb{R}^3)), \\ \Lambda_1^{5/4} u_3, \Lambda_2^{5/4} u_3, \nabla \Lambda_1^{5/4} u_3, \nabla \Lambda_2^{5/4} u_3 &\in L^2(0, \infty; L^2(\mathbb{R}^3)). \end{aligned} \quad (6)$$

此外, 如果有一个解在更好的函数空间, 则可以得到下面的唯一性结果.

定理 2 如果 $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ 是方程(5)的两个解, 它们都满足式(6)并且 $\nabla^{9/4} \mathbf{u}^{(2)} \in L^2(0, \infty; L^2(\mathbb{R}^3))$, 则 $\mathbf{u}^{(1)} \equiv \mathbf{u}^{(2)}$.

2 全局存在性和正则性的证明

首先给出用于证明上述定理的预备知识.

引理 1 (Sobolev 嵌入不等式) 假设 $2 \leq p \leq \infty$ 且 $s > d(1/2 - 1/p)$, 则存在一个仅依赖于 d, p 和 s 的常数 C , 使得

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}^{1 - \frac{d}{s}(\frac{1}{2} - \frac{1}{p})} \|\Lambda^s f\|_{L^2(\mathbb{R}^d)}^{\frac{d}{s}(\frac{1}{2} - \frac{1}{p})}, \quad \forall f \in H^s(\mathbb{R}^d), \quad (7)$$

此外, 如果 $2 \leq p < +\infty$, 式(7)对 $s = d(1/2 - 1/p)$ 也成立.

为了交换两个 Lebesgue 范数, 证明中将会使用下列 Minkowski 不等式和 Gagliardo-Nirenberg 不等式.

引理 2 (Minkowski 不等式) 假设 $f(x, y)$ 是一个测度函数, $x \in \mathbb{R}^m$ 且 $y \in \mathbb{R}^n$, 如果 $1 \leq q \leq p \leq \infty$, 则

$$\| \| f \|_{L_y^q(\mathbb{R}^n)} \|_{L_x^p(\mathbb{R}^m)} \leq \| \| f \|_{L_x^p(\mathbb{R}^m)} \|_{L_y^q(\mathbb{R}^n)}.$$

引理 3 (Gagliardo-Nirenberg 不等式) 如果 m, n, j 是非负整数且 $1 \leq p, q, r \leq \infty$, $\alpha \in [0, 1]$ 满足 $\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n} \right) \alpha + \frac{1-\alpha}{q}$ 和 $\frac{j}{m} \leq \alpha \leq 1$, 那么 u 的

第 j 阶弱导数属于 $L^p(\mathbb{R}^n)$ 空间, 且存在一个仅依赖于 m, n, j, p, q 和 r , 不依赖于 u 的常数 C , 使得

$$\|D^j u\|_{L^p} \leq C \|D^m u\|_{L^q}^\alpha \|u\|_{L^q}^{1-\alpha}.$$

下面证明解的全局存在性和正则性. 取方程(5)与 u 的内积, 分部积分并利用 $\nabla \cdot u = 0$ 得

$$\begin{aligned} & \|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^{\frac{5}{4}} u_1\|_{L^2}^2 d\tau + \\ & 2\nu \int_0^t \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_2\|_{L^2}^2 d\tau + \\ & 2\nu \int_0^t \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_3\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2, \end{aligned}$$

其中使用了如下记号:

$$\|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_2\|_{L^2}^2 := \|\Lambda_1^{\frac{5}{4}} u_2\|_{L^2}^2 + \|\Lambda_2^{\frac{5}{4}} u_2\|_{L^2}^2.$$

然后证明 $\|\nabla u\|_{L^2}$ 的全局有界性. 将 ∇ 应用于方程(5), 将所得方程与 ∇u 进行内积, 得到

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \nu \|\nabla \Lambda^{\frac{5}{4}} u_1\|_{L^2}^2 + \nu \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla u_2\|_{L^2}^2 + \\ & \nu \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla u_3\|_{L^2}^2 = I, \end{aligned}$$

其中

$$I = -\int \nabla(u \cdot \nabla u) \cdot \nabla u dx.$$

在利用散度条件 $\nabla \cdot u = 0$ 后, 可以将 I 写为

$$\begin{aligned} I &= -\int \partial_i u_j \partial_j u_k \partial_i u_k dx = -\sum_{i,j=1}^3 \int \partial_i u_j \partial_j u_1 \partial_i u_1 dx - \\ & \sum_{i,j=1}^3 \int \partial_i u_j \partial_j u_2 \partial_i u_2 dx - \sum_{i,j=1}^3 \int \partial_i u_j \partial_j u_3 \partial_i u_3 dx := \\ & I_1 + I_2 + I_3. \end{aligned}$$

由于对 u_1 及其导数有更多的控制, 所以在此只给出 I_2 和 I_3 估计的计算. 首先考虑 I_2 ,

$$\begin{aligned} I_2 &= -\int ((\partial_1 u_2)^2 \partial_1 u_1 + (\partial_1 u_2)^2 \partial_2 u_2 + \\ & \partial_1 u_3 \partial_3 u_2 \partial_1 u_2 + \partial_2 u_1 \partial_1 u_2 \partial_2 u_2 + (\partial_2 u_2)^3 + \\ & \partial_2 u_3 \partial_3 u_2 \partial_2 u_2 + \partial_3 u_1 \partial_1 u_2 \partial_3 u_2 + \\ & (\partial_3 u_2)^2 \partial_2 u_2 + (\partial_3 u_2)^2 \partial_3 u_3) dx, \end{aligned} \quad (8)$$

将方程(8)中的项根据它们出现的顺序依次标记为 $I_{21}, I_{22}, \dots, I_{29}$.

针对 I_{21} , 通过分部积分有

$$I_{21} = 2 \int u_1 \partial_1^2 u_2 \partial_1 u_2 dx.$$

利用 Hölder 不等式和引理 2 得

$$|I_{21}| \leq 2 \|u_1\|_{L_{\eta_1 \eta_2}^2 L_{\eta_3}^\infty} \|\partial_1^2 u_2\|_{L_{\eta_1 \eta_3}^2 L_{\eta_2}^4} \|\partial_1 u_2\|_{L_{\eta_3}^\infty L_{\eta_2}^4}. \quad (9)$$

由 Sobolev 不等式可得

$$\|u_1\|_{L_{\eta_1 \eta_2}^2 L_{\eta_3}^\infty} \leq C \|u_1\|_{L^2}^{\frac{3}{5}} \|\Lambda_3^{\frac{5}{4}} u_1\|_{L^2}^{\frac{2}{5}} \quad (10)$$

和

$$\|\partial_1^2 u_2\|_{L_{\eta_1 \eta_3}^2 L_{\eta_2}^4} \leq C \|\Lambda_2^{\frac{1}{4}} \partial_1^2 u_2\|_{L^2}. \quad (11)$$

由基本不等式

$$\begin{aligned} |\xi_2|^{\frac{1}{4}} |\xi_1| &\leq \frac{4}{5} |\xi_1|^{\frac{5}{4}} + \frac{1}{5} |\xi_2|^{\frac{5}{4}}, \\ |\xi_1|^{\frac{1}{4}} |\xi_2| &\leq \frac{1}{5} |\xi_1|^{\frac{5}{4}} + \frac{4}{5} |\xi_2|^{\frac{5}{4}} \end{aligned}$$

和 Plancherel 定理, 有

$$\|\Lambda_2^{\frac{1}{4}} \partial_1^2 u_2\|_{L^2}^2 \leq C (\|\Lambda_1^{\frac{5}{4}} \partial_1 u_2\|_{L^2}^2 + \|\Lambda_2^{\frac{5}{4}} \partial_1 u_2\|_{L^2}^2). \quad (12)$$

应用引理 1 和引理 3, 取 $p = \infty$ 和 $s = 1$, 有

$$\begin{aligned} \|\partial_1 u_2\|_{L_{\eta_3}^2 L_{\eta_1}^\infty L_{\eta_2}^4} &\leq C \|\Lambda_2^{\frac{1}{4}} \partial_1 u_2\|_{L_{\eta_2 \eta_3}^\infty L_{\eta_1}^\infty} \leq \\ & C \|\Lambda_2^{\frac{1}{4}} \partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\Lambda_1 \Lambda_2^{\frac{1}{4}} \partial_1 u_2\|_{L^2}^{\frac{1}{2}} \leq \\ & C \|\partial_1 u_2\|_{L^2}^{\frac{2}{5}} \|\Lambda_2^{\frac{5}{4}} \partial_1 u_2\|_{L^2}^{\frac{1}{5}} \|\Lambda_1 \Lambda_2^{\frac{1}{4}} \partial_1 u_2\|_{L^2}^{\frac{1}{2}} \leq \\ & C \|\partial_1 u_2\|_{L^2}^{\frac{2}{5}} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_1 u_2\|_{L^2}^{\frac{3}{5}}, \end{aligned} \quad (13)$$

其中使用了插值不等式

$$\|\Lambda_2^{\frac{1}{4}} \partial_1 u_2\|_{L_{\eta_2}^2} \leq C \|\partial_1 u_2\|_{L_{\eta_2}^5}^{\frac{4}{5}} \|\Lambda_2^{\frac{5}{4}} \partial_1 u_2\|_{L_{\eta_2}^2}^{\frac{1}{5}}.$$

综合估计式(9)~(13)并使用 Young 不等式, 有

$$\begin{aligned} |I_{21}| &\leq C \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_1 u_2\|_{L^2}^{\frac{8}{5}} \|u_1\|_{L^2}^{\frac{3}{5}} \|\Lambda_3^{\frac{5}{4}} u_1\|_{L^2}^{\frac{2}{5}} \times \\ & \|\partial_1 u_2\|_{L^2}^{\frac{2}{5}} \leq \frac{\nu}{2048} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_1 u_2\|_{L^2}^2 + \\ & C \|u_1\|_{L^2}^3 \|\Lambda_3^{\frac{5}{4}} u_1\|_{L^2}^2 \|\partial_1 u_2\|_{L^2}^2. \end{aligned}$$

对于 I_{22} 的估计, 可以利用类似于 I_{21} 估计的方法. 通过分部积分法有

$$I_{22} = -\int \partial_2 u_2 (\partial_1 u_2)^2 dx = 2 \int u_2 \partial_1 \partial_2 u_2 \partial_1 u_2 dx,$$

事实上,

$$\begin{aligned} |I_{22}| &\leq C \|\partial_1 \partial_2 u_2\|_{L_{\eta_1 \eta_3}^2 L_{\eta_2}^4} \|u_2\|_{L_{\eta_3}^2 L_{\eta_1}^\infty L_{\eta_2}^4} \|\partial_1 u_2\|_{L_{\eta_3}^\infty L_{\eta_1}^4} \leq \\ & C \|\Lambda_2^{\frac{1}{4}} \partial_1 \partial_2 u_2\|_{L^2} \|u_2\|_{L^2}^{\frac{1}{2}} \|\nabla_h u_2\|_{L^2}^{\frac{1}{2}} \|\Lambda_1^{\frac{1}{4}} \partial_1 u_2\|_{L^2}^{\frac{1}{2}} \times \\ & \|\Lambda_3 \Lambda_1^{\frac{1}{4}} \partial_1 u_2\|_{L^2}^{\frac{1}{2}} \leq \frac{\nu}{2048} \|\Lambda_2^{\frac{5}{4}} \partial_1 u_2\|_{L^2}^2 + \frac{\nu}{2048} \times \\ & \|\Lambda_1^{\frac{5}{4}} \partial_3 u_2\|_{L^2}^2 + C \|u_2\|_{L^2}^2 \|\Lambda_1^{\frac{5}{4}} u_2\|_{L^2}^2 \|\nabla_h u_2\|_{L^2}^2, \end{aligned}$$

其中 $\nabla_h = (\partial_1, \partial_2)$.

针对 I_{23} 估计, 通过分部积分有

$$I_{23} = -\int \partial_1 u_3 \partial_3 u_2 \partial_1 u_2 dx = \int u_2 \partial_1 \partial_3 u_3 \partial_1 u_2 dx + \int u_2 \partial_1 u_3 \partial_1 \partial_3 u_2 dx := I_{231} + I_{232}.$$

事实上,

$$\begin{aligned} I_{231} &= \int u_2 \partial_1 \partial_3 u_3 \partial_1 u_2 dx \leq \| \partial_1 \partial_3 u_3 \|_{L_{x_1 x_3}^2 L_{x_2}^4} \| u_2 \|_{L_{x_3}^2 L_{x_1 x_2}^4} \times \\ &\| \partial_1 u_2 \|_{L_{x_2}^2 L_{x_3}^{\infty} L_{x_1}^4} \leq C \| \Lambda_2^{\frac{1}{4}} \partial_1 \partial_3 u_3 \|_{L^2} \| u_2 \|_{L^2}^{\frac{1}{2}} \times \\ &\| \nabla_h u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_1^{\frac{1}{4}} \partial_1 u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_1^{\frac{1}{4}} \partial_1 u_2 \|_{L^2}^{\frac{1}{2}} \leq \\ &\frac{\nu}{2048} \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_3 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_3 u_2 \|_{L^2}^2 + \\ &C \| u_2 \|_{L^2}^2 \| \Lambda_1^{\frac{5}{4}} u_2 \|_{L^2}^2 \| \nabla_h u_2 \|_{L^2}^2. \end{aligned}$$

类似地,

$$\begin{aligned} I_{232} &= \int u_1 \partial_1 u_3 \partial_1 \partial_3 u_2 dx \leq C \| \partial_1 \partial_3 u_2 \|_{L_{x_1 x_3}^2 L_{x_2}^4} \times \\ &\| u_2 \|_{L_{x_3}^2 L_{x_1 x_2}^4} \| \partial_1 u_3 \|_{L_{x_2}^2 L_{x_3}^{\infty} L_{x_1}^4} \leq C \| \Lambda_2^{\frac{1}{4}} \partial_1 \partial_3 u_2 \|_{L^2} \times \\ &\| u_2 \|_{L^2}^{\frac{1}{2}} \| \nabla_h u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_1^{\frac{1}{4}} \partial_1 u_3 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_1^{\frac{1}{4}} \partial_1 u_3 \|_{L^2}^{\frac{1}{2}} \leq \\ &\frac{\nu}{2048} \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_2 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_3 u_3 \|_{L^2}^2 + \\ &C \| u_2 \|_{L^2}^2 \| \Lambda_1^{\frac{5}{4}} u_3 \|_{L^2}^2 \| \nabla_h u_2 \|_{L^2}^2. \end{aligned}$$

这就确定了 I_{23} 的估计.

对于 I_{24} 的估计, 通过分部积分有

$$I_{24} = -\int \partial_2 u_1 \partial_1 u_2 \partial_2 u_2 dx = \int u_2 \partial_1 \partial_2 u_1 \partial_2 u_2 dx + \int u_2 \partial_2 u_1 \partial_1 \partial_2 u_2 dx := I_{241} + I_{242}.$$

事实上,

$$\begin{aligned} I_{241} &= \int u_2 \partial_1 \partial_2 u_1 \partial_2 u_2 dx \leq \| \partial_1 \partial_2 u_1 \|_{L_{x_2 x_3}^2 L_{x_1}^4} \| u_2 \|_{L_{x_3}^2 L_{x_1 x_2}^4} \times \\ &\| \partial_2 u_2 \|_{L_{x_1}^2 L_{x_3}^{\infty} L_{x_2}^4} \leq C \| \Lambda_1^{\frac{5}{4}} \partial_2 u_1 \|_{L^2} \| u_2 \|_{L^2}^{\frac{1}{2}} \times \\ &\| \nabla_h u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \leq \\ &\frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_2 u_1 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_3 u_2 \|_{L^2}^2 + \\ &C \| u_2 \|_{L^2}^2 \| \nabla_h u_2 \|_{L^2}^2 \| \Lambda_2^{\frac{5}{4}} u_2 \|_{L^2}^2. \end{aligned}$$

类似地,

$$\begin{aligned} I_{242} &= \int u_2 \partial_2 u_1 \partial_1 \partial_2 u_2 dx \leq \| \partial_1 \partial_2 u_2 \|_{L_{x_2 x_3}^2 L_{x_1}^4} \| u_2 \|_{L_{x_3}^2 L_{x_1 x_2}^4} \times \\ &\| \partial_2 u_1 \|_{L_{x_1}^2 L_{x_3}^{\infty} L_{x_2}^4} \leq C \| \Lambda_1^{\frac{5}{4}} \partial_2 u_2 \|_{L^2} \| u_2 \|_{L^2}^{\frac{1}{2}} \times \\ &\| \nabla_h u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_2^{\frac{1}{4}} \partial_2 u_1 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_1 \|_{L^2}^{\frac{1}{2}} \leq \end{aligned}$$

$$\frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_2 u_2 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_3 u_1 \|_{L^2}^2 +$$

$$C \| u_2 \|_{L^2}^2 \| \nabla_h u_2 \|_{L^2}^2 \| \Lambda_2^{\frac{5}{4}} u_1 \|_{L^2}^2.$$

这就完成了 I_{24} 的估计.

为了估计 I_{25} , 利用分部积分得到

$$I_{25} = 2 \int u_2 \partial_2 \partial_2 u_2 \partial_2 u_2 dx \leq 2 \| \partial_2 \partial_2 u_2 \|_{L_{x_2 x_3}^2 L_{x_1}^4} \times$$

$$\begin{aligned} &\| \partial_2 u_2 \|_{L_{x_1}^2 L_{x_3}^{\infty} L_{x_2}^4} \| u_2 \|_{L_{x_3}^2 L_{x_1 x_2}^4} \leq C \| \Lambda_1^{\frac{1}{4}} \partial_2 \partial_2 u_2 \|_{L^2} \times \\ &\| \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \| u_2 \|_{L^2}^{\frac{1}{2}} \| \nabla_h u_2 \|_{L^2}^{\frac{1}{2}} \leq \\ &\frac{\nu}{2048} \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_2 u_2 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_3 u_2 \|_{L^2}^2 + \end{aligned}$$

$$C \| u_2 \|_{L^2}^2 \| \Lambda_2^{\frac{5}{4}} u_2 \|_{L^2}^2 \| \nabla_h u_2 \|_{L^2}^2.$$

对 I_{26} 的估计与 I_{24} 类似. 通过分部积分有

$$I_{26} = -\int \partial_2 u_3 \partial_3 u_2 \partial_2 u_2 dx = \int u_2 \partial_2 \partial_3 u_3 \partial_2 u_2 dx + \int u_2 \partial_2 u_3 \partial_2 \partial_3 u_2 dx := I_{261} + I_{262}.$$

事实上,

$$I_{261} = \int u_2 \partial_2 \partial_3 u_3 \partial_2 u_2 dx \leq \| \partial_2 \partial_3 u_3 \|_{L_{x_2 x_3}^2 L_{x_1}^4} \times$$

$$\begin{aligned} &\| u_2 \|_{L_{x_3}^2 L_{x_1 x_2}^4} \| \partial_2 u_2 \|_{L_{x_1}^2 L_{x_3}^{\infty} L_{x_2}^4} \leq C \| \Lambda_1^{\frac{1}{4}} \partial_2 \partial_3 u_3 \|_{L^2} \times \\ &\| u_2 \|_{L^2}^{\frac{1}{2}} \| \nabla_h u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \leq \\ &\frac{\nu}{2048} \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_3 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_3 u_2 \|_{L^2}^2 + \\ &C \| u_2 \|_{L^2}^2 \| \nabla_h u_2 \|_{L^2}^2 \| \Lambda_2^{\frac{5}{4}} u_2 \|_{L^2}^2. \end{aligned}$$

同理得

$$I_{262} = \int u_2 \partial_2 u_3 \partial_2 \partial_3 u_2 dx \leq \| \partial_2 \partial_3 u_2 \|_{L_{x_2 x_3}^2 L_{x_1}^4} \times$$

$$\begin{aligned} &\| u_2 \|_{L_{x_3}^2 L_{x_1 x_2}^4} \| \partial_2 u_3 \|_{L_{x_1}^2 L_{x_3}^{\infty} L_{x_2}^4} \leq C \| \Lambda_1^{\frac{1}{4}} \partial_2 \partial_3 u_2 \|_{L^2} \times \\ &\| u_2 \|_{L^2}^{\frac{1}{2}} \| \nabla_h u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_2^{\frac{1}{4}} \partial_2 u_3 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_3 \|_{L^2}^{\frac{1}{2}} \leq \\ &\frac{\nu}{2048} \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_2 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_3 u_3 \|_{L^2}^2 + \\ &C \| u_2 \|_{L^2}^2 \| \nabla_h u_2 \|_{L^2}^2 \| \Lambda_2^{\frac{5}{4}} u_3 \|_{L^2}^2. \end{aligned}$$

因此就得到了 I_{26} 的估计.

为估计 I_{27} , 通过分部积分可以得到

$$I_{27} = -\int \partial_3 u_1 \partial_1 u_2 \partial_3 u_2 dx = \int u_2 \partial_3 \partial_3 u_1 \partial_1 u_2 dx + \int u_2 \partial_3 u_1 \partial_1 \partial_3 u_2 dx := I_{271} + I_{272}.$$

对 I_{271} 的估计与 I_{22} 的估计类似, 因此有

$$\begin{aligned} I_{271} &= \int u_2 \partial_3 \partial_3 u_1 \partial_1 u_2 dx \leq \| \partial_3 \partial_3 u_1 \|_{L^2_{x_1 x_2} L^4_{x_3}} \times \\ &\quad \| \partial_1 u_2 \|_{L^2_{x_3} L^\infty_{x_1} L^4_{x_2}} \| u_2 \|_{L^2_{x_2} L^4_{x_1 x_3}} \leq C \| \Lambda_3^{\frac{5}{4}} \partial_3 u_1 \|_{L^2} \times \\ &\quad \| \Lambda_1^{\frac{1}{4}} \partial_1 u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_1^{\frac{1}{4}} \partial_2 \partial_1 u_2 \|_{L^2}^{\frac{1}{2}} \| u_2 \|_{L^2}^{\frac{1}{2}} \times \\ &\quad \| (\partial_1, \partial_3) u_2 \|_{L^2}^{\frac{1}{2}} \leq \frac{\nu}{2048} \| \Lambda_3^{\frac{5}{4}} \partial_3 u_1 \|_{L^2}^2 + \frac{\nu}{2048} \times \\ &\quad \| \Lambda_1^{\frac{5}{4}} \partial_2 u_2 \|_{L^2}^2 + C \| u_2 \|_{L^2}^2 \| \Lambda_1^{\frac{5}{4}} u_2 \|_{L^2}^2 \| (\partial_1, \partial_3) u_2 \|_{L^2}^2. \end{aligned}$$

类似地,

$$\begin{aligned} I_{272} &= \int u_2 \partial_3 u_1 \partial_1 \partial_3 u_2 dx \leq \| \partial_1 \partial_3 u_2 \|_{L^2_{x_2 x_3} L^4_{x_1}} \times \\ &\quad \| \partial_3 u_1 \|_{L^2_{x_1} L^\infty_{x_2} L^4_{x_3}} \| u_2 \|_{L^2_{x_2} L^4_{x_1 x_3}} \leq C \| \Lambda_1^{\frac{5}{4}} \partial_3 u_2 \|_{L^2} \times \\ &\quad \| \Lambda_3^{\frac{1}{4}} \partial_3 u_1 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3^{\frac{1}{4}} \partial_2 \partial_3 u_1 \|_{L^2}^{\frac{1}{2}} \| u_2 \|_{L^2}^{\frac{1}{2}} \times \\ &\quad \| (\partial_1, \partial_3) u_2 \|_{L^2}^{\frac{1}{2}} \leq \frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_3 u_2 \|_{L^2}^2 + \\ &\quad \frac{\nu}{2048} \| (\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \partial_3 u_1 \|_{L^2}^2 + \\ &\quad C \| u_2 \|_{L^2}^2 \| (\partial_1, \partial_3) u_2 \|_{L^2}^2 \| \Lambda_3^{\frac{5}{4}} u_1 \|_{L^2}^2. \end{aligned}$$

因此就完成了 I_{27} 的估计.

现在转向 I_{28} 和 I_{29} . 通过分部积分和散度条件 $\nabla \cdot u = 0$ 得

$$\begin{aligned} I_{28} + I_{29} &= - \int [(\partial_3 u_2)^2 \partial_2 u_2 + (\partial_3 u_2)^2 \partial_3 u_3] dx = \\ &= \int (\partial_3 u_2)^2 \partial_1 u_1 dx = -2 \int u_1 \partial_3 u_2 \partial_1 \partial_3 u_2 dx := I_{20}. \end{aligned}$$

对 I_{20} 的估计与 I_{21} 类似. 因此可得

$$\begin{aligned} |I_{20}| &\leq \| \partial_1 \partial_3 u_2 \|_{L^2_{x_2 x_3} L^4_{x_1}} \| u_1 \|_{L^2_{x_1 x_2} L^4_{x_3}} \| \partial_3 u_2 \|_{L^2_{x_3} L^\infty_{x_2} L^4_{x_1}} \leq \\ &\quad C \| \Lambda_1^{\frac{5}{4}} \partial_3 u_2 \|_{L^2} \| u_1 \|_{L^2}^{\frac{3}{2}} \| \Lambda_3^{\frac{5}{4}} u_1 \|_{L^2}^{\frac{2}{2}} \| \partial_3 u_2 \|_{L^2}^{\frac{2}{2}} \times \\ &\quad \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_2 \|_{L^2}^{\frac{3}{2}} \leq \frac{\nu}{2048} \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_2 \|_{L^2}^2 + \\ &\quad C \| u_1 \|_{L^2}^3 \| \Lambda_3^{\frac{5}{4}} u_1 \|_{L^2}^2 \| \partial_3 u_2 \|_{L^2}^2. \end{aligned}$$

这就完成了 I_2 的估计.

现在考虑 I_3 . 注意 $\partial_3 u_3$ 出现在 I_3 的许多项中, 通过散度条件可以将其转换为 $-(\partial_1 u_1 + \partial_2 u_2)$. 更准确地说,

$$\begin{aligned} I_3 &= - \int ((\partial_1 u_3)^2 \partial_1 u_1 + \partial_1 u_2 \partial_2 u_3 \partial_1 u_3 + \\ &\quad (\partial_1 u_3)^2 \partial_3 u_3 + \partial_2 u_1 \partial_1 u_3 \partial_2 u_3 + (\partial_2 u_3)^2 \partial_2 u_2 + \\ &\quad (\partial_2 u_3)^2 \partial_3 u_3 + \partial_3 u_1 \partial_1 u_3 \partial_3 u_3 + \end{aligned}$$

$$\partial_3 u_2 \partial_2 u_3 \partial_3 u_3 + (\partial_3 u_3)^3) dx. \quad (14)$$

将方程(14)中的项根据它们出现的顺序依次标记为 $I_{31}, I_{32}, \dots, I_{39}$.

首先处理 I_{31} .

$$\begin{aligned} I_{31} &= 2 \int u_1 \partial_1 \partial_1 u_3 \partial_1 u_3 dx \leq 2 \| \partial_1 \partial_1 u_3 \|_{L^2_{x_1 x_3} L^4_{x_2}} \times \\ &\quad \| \partial_1 u_3 \|_{L^2_{x_3} L^\infty_{x_1} L^4_{x_2}} \| u_1 \|_{L^2_{x_2} L^\infty_{x_1} L^4_{x_3}} \leq C \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_1 u_3 \|_{L^2} \times \\ &\quad \| \Lambda_2^{\frac{1}{4}} \partial_1 u_3 \|_{L^2}^{\frac{1}{2}} \| \Lambda_1 \Lambda_2^{\frac{1}{4}} \partial_1 u_3 \|_{L^2}^{\frac{1}{2}} \| u_1 \|_{L^2}^{\frac{3}{2}} \| \Lambda_3^{\frac{5}{4}} u_1 \|_{L^2}^{\frac{2}{2}} \leq \\ &\quad C \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_1 u_3 \|_{L^2} \| \partial_1 u_3 \|_{L^2}^{\frac{5}{2}} \| \Lambda_2^{\frac{5}{4}} \partial_1 u_3 \|_{L^2}^{\frac{1}{2}} \times \\ &\quad \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_1 u_3 \|_{L^2}^{\frac{1}{2}} \| u_1 \|_{L^2}^{\frac{3}{2}} \| \Lambda_3^{\frac{5}{4}} u_1 \|_{L^2}^{\frac{2}{2}} \leq \frac{\nu}{2048} \times \\ &\quad \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_1 u_3 \|_{L^2}^2 + C \| u_1 \|_{L^2}^3 \| \Lambda_3^{\frac{5}{4}} u_1 \|_{L^2}^2 \| \partial_1 u_3 \|_{L^2}^2. \end{aligned}$$

通过分部积分有

$$\begin{aligned} I_{32} &= - \int \partial_1 u_2 \partial_2 u_3 \partial_1 u_3 dx = \int u_3 \partial_1 \partial_2 u_2 \partial_1 u_3 dx + \\ &= \int u_3 \partial_1 u_2 \partial_1 \partial_2 u_3 dx := I_{321} + I_{322}. \end{aligned}$$

对 I_{321} 的估计与 I_{22} 的估计类似, 因此有

$$\begin{aligned} I_{321} &= \int u_3 \partial_1 \partial_2 u_2 \partial_1 u_3 dx \leq \| \partial_1 \partial_2 u_2 \|_{L^2_{x_1 x_3} L^4_{x_2}} \| u_3 \|_{L^2_{x_3} L^4_{x_1 x_2}} \times \\ &\quad \| \partial_1 u_3 \|_{L^2_{x_2} L^\infty_{x_1} L^4_{x_3}} \leq C \| \Lambda_2^{\frac{1}{4}} \partial_1 \partial_2 u_2 \|_{L^2} \| u_3 \|_{L^2}^{\frac{1}{2}} \times \\ &\quad \| \nabla_h u_3 \|_{L^2}^{\frac{1}{2}} \| \Lambda_1^{\frac{1}{4}} \partial_1 u_3 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_1^{\frac{1}{4}} \partial_1 u_3 \|_{L^2}^{\frac{1}{2}} \leq \\ &\quad \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_1 u_2 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_3 u_3 \|_{L^2}^2 + \\ &\quad C \| u_3 \|_{L^2}^2 \| \Lambda_1^{\frac{5}{4}} u_3 \|_{L^2}^2 \| \nabla_h u_3 \|_{L^2}^2. \end{aligned}$$

类似地,

$$\begin{aligned} I_{322} &= \int u_3 \partial_1 \partial_2 u_3 \partial_1 u_2 dx \leq \| \partial_1 \partial_2 u_3 \|_{L^2_{x_1 x_3} L^4_{x_2}} \times \\ &\quad \| u_3 \|_{L^2_{x_3} L^4_{x_1 x_2}} \| \partial_1 u_2 \|_{L^2_{x_2} L^\infty_{x_1} L^4_{x_3}} \leq C \| \Lambda_2^{\frac{1}{4}} \partial_1 \partial_2 u_3 \|_{L^2} \times \\ &\quad \| u_3 \|_{L^2}^{\frac{1}{2}} \| \nabla_h u_3 \|_{L^2}^{\frac{1}{2}} \| \Lambda_1^{\frac{1}{4}} \partial_1 u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_1^{\frac{1}{4}} \partial_1 u_2 \|_{L^2}^{\frac{1}{2}} \leq \\ &\quad \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_1 u_3 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_3 u_2 \|_{L^2}^2 + \\ &\quad C \| u_3 \|_{L^2}^2 \| \Lambda_1^{\frac{5}{4}} u_2 \|_{L^2}^2 \| \nabla_h u_3 \|_{L^2}^2. \end{aligned}$$

用估计 I_{321} 的方法去估计 I_{33} 得

$$\begin{aligned} I_{33} &= 2 \int u_3 \partial_1 \partial_3 u_3 \partial_1 u_3 dx \leq \| \partial_1 \partial_3 u_3 \|_{L^2_{x_1 x_3} L^4_{x_2}} \| u_3 \|_{L^2_{x_3} L^4_{x_1 x_2}} \times \\ &\quad \| \partial_1 u_3 \|_{L^2_{x_2} L^\infty_{x_1} L^4_{x_3}} \leq C \| \Lambda_2^{\frac{1}{4}} \partial_1 \partial_3 u_3 \|_{L^2} \| u_3 \|_{L^2}^{\frac{1}{2}} \times \end{aligned}$$

$$\begin{aligned} & \|\nabla_h u_3\|_{L^2}^2 \|\Lambda_1^{\frac{1}{4}} \partial_1 u_3\|_{L^2}^2 \|\Lambda_3 \Lambda_1^{\frac{1}{4}} \partial_1 u_3\|_{L^2}^2 \leq \\ & \frac{\nu}{2048} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_3\|_{L^2}^2 + \frac{\nu}{2048} \|\Lambda_1^{\frac{5}{4}} \partial_3 u_3\|_{L^2}^2 + \\ & C \|u_3\|_{L^2}^2 \|\Lambda_1^{\frac{5}{4}} u_3\|_{L^2}^2 \|\nabla_h u_3\|_{L^2}^2. \end{aligned}$$

通过分部积分可得

$$\begin{aligned} I_{34} &= -\int \partial_2 u_1 \partial_1 u_3 \partial_2 u_3 dx = \int u_3 \partial_1 \partial_2 u_1 \partial_2 u_3 dx + \\ & \int u_3 \partial_2 u_1 \partial_1 \partial_2 u_3 dx := I_{341} + I_{342}. \end{aligned}$$

对 I_{341} 的估计跟 I_{331} 的估计方法类似, 因此有

$$\begin{aligned} I_{341} &= \int u_3 \partial_1 \partial_2 u_1 \partial_2 u_3 dx \leq \|\partial_1 \partial_2 u_1\|_{L_{x_2 x_3}^2 L_{x_1}^4} \times \\ & \|u_3\|_{L_{x_3}^2 L_{x_1 x_2}^4} \|\partial_2 u_3\|_{L_{x_1}^2 L_{x_3}^\infty L_{x_2}^4} \leq C \|\Lambda_1^{\frac{5}{4}} \partial_2 u_1\|_{L^2} \times \\ & \|u_3\|_{L^2}^{\frac{1}{2}} \|\nabla_h u_3\|_{L^2}^{\frac{1}{2}} \|\Lambda_2^{\frac{1}{4}} \partial_2 u_3\|_{L^2}^{\frac{1}{2}} \|\Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_3\|_{L^2}^{\frac{1}{2}} \leq \\ & \frac{\nu}{2048} \|\Lambda_1^{\frac{5}{4}} \partial_2 u_1\|_{L^2}^2 + \frac{\nu}{2048} \|\Lambda_2^{\frac{5}{4}} \partial_3 u_3\|_{L^2}^2 + \\ & C \|u_3\|_{L^2}^2 \|\nabla_h u_3\|_{L^2}^2 \|\Lambda_2^{\frac{5}{4}} u_3\|_{L^2}^2. \end{aligned}$$

同理得

$$\begin{aligned} I_{342} &= \int u_3 \partial_1 \partial_2 u_3 \partial_2 u_1 dx \leq \|\partial_1 \partial_2 u_3\|_{L_{x_2 x_3}^2 L_{x_1}^4} \|u_3\|_{L_{x_3}^2 L_{x_1 x_2}^4} \times \\ & \|\partial_2 u_1\|_{L_{x_1}^2 L_{x_3}^\infty L_{x_2}^4} \leq C \|\Lambda_1^{\frac{5}{4}} \partial_2 u_3\|_{L^2} \|u_3\|_{L^2}^{\frac{1}{2}} \times \\ & \|\nabla_h u_3\|_{L^2}^{\frac{1}{2}} \|\Lambda_2^{\frac{1}{4}} \partial_2 u_1\|_{L^2}^{\frac{1}{2}} \|\Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_1\|_{L^2}^{\frac{1}{2}} \leq \\ & \frac{\nu}{2048} \|\Lambda_1^{\frac{5}{4}} \partial_2 u_3\|_{L^2}^2 + \frac{\nu}{2048} \|\Lambda_2^{\frac{5}{4}} \partial_3 u_3\|_{L^2}^2 + \\ & C \|u_3\|_{L^2}^2 \|\nabla_h u_3\|_{L^2}^2 \|\Lambda_2^{\frac{5}{4}} u_1\|_{L^2}^2. \end{aligned}$$

现在转向 I_{35} 和 I_{36} . 根据分部积分和散度条件

$\nabla \cdot u = 0$ 有

$$\begin{aligned} I_{35} + I_{36} &= -\int [(\partial_2 u_3)^2 \partial_2 u_2 + (\partial_2 u_3)^2 \partial_3 u_3] dx = \\ &= \int (\partial_2 u_3)^2 \partial_1 u_1 dx := I_{30}. \end{aligned}$$

通过分部积分可得

$$I_{30} = \int (\partial_2 u_3)^2 \partial_1 u_1 dx = -2 \int u_1 \partial_1 \partial_2 u_3 \partial_2 u_3 dx,$$

因此有

$$\begin{aligned} |I_{30}| &\leq C \|\partial_1 \partial_2 u_3\|_{L_{x_2 x_3}^2 L_{x_1}^4} \|u_1\|_{L_{x_1 x_2}^2 L_{x_3}^\infty} \|\partial_2 u_3\|_{L_{x_3}^2 L_{x_2}^\infty L_{x_1}^4} \leq \\ & C \|\Lambda_1^{\frac{5}{4}} \partial_2 u_3\|_{L^2} \|u_1\|_{L^2}^{\frac{3}{2}} \|\Lambda_3 u_1\|_{L^2}^{\frac{2}{5}} \|\partial_2 u_3\|_{L^2}^{\frac{2}{5}} \times \\ & \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_2 u_3\|_{L^2}^{\frac{3}{5}} \leq \frac{\nu}{2048} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_2 u_3\|_{L^2}^2 + \\ & C \|u_1\|_{L^2}^3 \|\Lambda_3 u_1\|_{L^2}^2 \|\partial_2 u_3\|_{L^2}^2. \end{aligned}$$

考虑 I_{37} , 通过分部积分得

$$\begin{aligned} I_{37} &= -\int \partial_3 u_1 \partial_1 u_3 \partial_3 u_3 dx = \int u_3 \partial_3 \partial_3 u_1 \partial_1 u_3 dx + \\ & \int u_3 \partial_3 u_1 \partial_1 \partial_3 u_3 dx := I_{371} + I_{372}. \end{aligned}$$

事实上,

$$\begin{aligned} I_{371} &= \int u_3 \partial_3 \partial_3 u_1 \partial_1 u_3 dx \leq \|\partial_3 \partial_3 u_1\|_{L_{x_1 x_2}^2 L_{x_3}^4} \times \\ & \|\partial_1 u_3\|_{L_{x_3}^2 L_{x_2}^\infty L_{x_1}^4} \|u_3\|_{L_{x_2}^2 L_{x_1 x_3}^4} \leq C \|\Lambda_3^{\frac{5}{4}} \partial_3 u_1\|_{L^2} \times \\ & \|\Lambda_1^{\frac{1}{4}} \partial_1 u_3\|_{L^2}^{\frac{1}{2}} \|\Lambda_1^{\frac{1}{4}} \partial_2 \partial_1 u_3\|_{L^2}^{\frac{1}{2}} \|u_3\|_{L^2}^{\frac{1}{2}} \|(\partial_1, \partial_3) u_3\|_{L^2}^{\frac{1}{2}} \leq \\ & \frac{\nu}{2048} \|\Lambda_3^{\frac{5}{4}} \partial_3 u_1\|_{L^2}^2 + \frac{\nu}{2048} \|\Lambda_1^{\frac{5}{4}} \partial_2 u_3\|_{L^2}^2 + \\ & C \|u_3\|_{L^2}^2 \|\Lambda_1^{\frac{5}{4}} u_3\|_{L^2}^2 \|(\partial_1, \partial_3) u_3\|_{L^2}^2. \end{aligned}$$

类似地,

$$\begin{aligned} I_{372} &= \int u_3 \partial_3 u_1 \partial_1 \partial_3 u_3 dx \leq \|\partial_1 \partial_3 u_3\|_{L_{x_2 x_3}^2 L_{x_1}^4} \times \\ & \|\partial_3 u_1\|_{L_{x_1}^2 L_{x_2}^\infty L_{x_3}^4} \|u_3\|_{L_{x_2}^2 L_{x_1 x_3}^4} \leq C \|\Lambda_1^{\frac{5}{4}} \partial_3 u_3\|_{L^2} \times \\ & \|\Lambda_3^{\frac{1}{4}} \partial_3 u_1\|_{L^2}^{\frac{1}{2}} \|\Lambda_3^{\frac{1}{4}} \partial_2 \partial_3 u_1\|_{L^2}^{\frac{1}{2}} \|u_3\|_{L^2}^{\frac{1}{2}} \|(\partial_1, \partial_3) u_3\|_{L^2}^{\frac{1}{2}} \leq \\ & \frac{\nu}{2048} \|\Lambda_1^{\frac{5}{4}} \partial_3 u_3\|_{L^2}^2 + \frac{\nu}{2048} \|\Lambda_3^{\frac{5}{4}} \partial_2 u_1\|_{L^2}^2 + \\ & C \|u_3\|_{L^2}^2 \|(\partial_1, \partial_3) u_3\|_{L^2}^2 \|\Lambda_3^{\frac{5}{4}} u_1\|_{L^2}^2. \end{aligned}$$

根据 $\nabla \cdot u = 0$ 可得

$$\begin{aligned} I_{38} &= -\int \partial_3 u_2 \partial_2 u_3 \partial_3 u_3 dx = \\ & -\int \partial_3 u_2 \partial_2 u_3 (-\partial_1 u_1 - \partial_2 u_2) dx = \\ & \int \partial_3 u_2 \partial_2 u_3 \partial_1 u_1 dx + \int \partial_3 u_2 \partial_2 u_3 \partial_2 u_2 dx := \\ & I_{381} + I_{382}. \end{aligned}$$

用分部积分可以计算出

$$\begin{aligned} I_{381} &= \int \partial_3 u_2 \partial_2 u_3 \partial_1 u_1 dx = \int u_2 \partial_2 \partial_3 u_3 \partial_1 u_1 dx + \\ & \int u_2 \partial_2 u_3 \partial_1 \partial_3 u_1 dx := I_{3811} + I_{3812}, \end{aligned}$$

其中

$$\begin{aligned} I_{3811} &= \int u_2 \partial_2 \partial_3 u_3 \partial_1 u_1 dx \leq \|\partial_2 \partial_3 u_3\|_{L_{x_1 x_3}^2 L_{x_2}^4} \|u_2\|_{L_{x_3}^2 L_{x_1 x_2}^4} \times \\ & \|\partial_1 u_1\|_{L_{x_2}^2 L_{x_3}^\infty L_{x_1}^4} \leq C \|\Lambda_2^{\frac{5}{4}} \partial_3 u_3\|_{L^2} \|u_2\|_{L^2}^{\frac{1}{2}} \times \\ & \|\nabla_h u_2\|_{L^2}^{\frac{1}{2}} \|\Lambda_1^{\frac{1}{4}} \partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\Lambda_3 \Lambda_1^{\frac{1}{4}} \partial_1 u_1\|_{L^2}^{\frac{1}{2}} \leq \\ & \frac{\nu}{2048} \|\Lambda_2^{\frac{5}{4}} \partial_3 u_3\|_{L^2}^2 + \frac{\nu}{2048} \|\Lambda_1^{\frac{5}{4}} \partial_3 u_1\|_{L^2}^2 + \\ & C \|u_2\|_{L^2}^2 \|\nabla_h u_2\|_{L^2}^2 \|\Lambda_1^{\frac{5}{4}} u_1\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned}
I_{3812} &= \int u_2 \partial_2 u_3 \partial_1 \partial_3 u_1 dx \leq \| \partial_1 \partial_3 u_1 \|_{L^2_{x_2 x_3} L^4_{x_1}} \| u_2 \|_{L^2_{x_3} L^4_{x_1 x_2}} \times \\
&\| \partial_2 u_3 \|_{L^2_{x_1} L^\infty_{x_3} L^4_{x_2}} \leq C \| \Lambda_1^{\frac{5}{4}} \partial_3 u_1 \|_{L^2} \| u_2 \|_{L^2}^{\frac{1}{2}} \times \\
&\| \nabla_h u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_2^{\frac{1}{4}} \partial_2 u_3 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_3 \|_{L^2}^{\frac{1}{2}} \leq \\
&\frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_3 u_1 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_3 u_3 \|_{L^2}^2 + \\
&C \| u_2 \|_{L^2}^2 \| \nabla_h u_2 \|_{L^2}^2 \| \Lambda_2^{\frac{5}{4}} u_3 \|_{L^2}^2.
\end{aligned}$$

由于 $I_{382} = -I_{22}$, 因此省略.

根据散度条件 $\nabla \cdot u = 0$, 可以计算出

$$\begin{aligned}
I_{39} &= -\int (\partial_3 u_3)^3 dx = -\int (\partial_3 u_3)^2 (-\partial_1 u_1 - \partial_2 u_2) dx = \\
&\int (\partial_3 u_3)^2 \partial_1 u_1 dx + \int (\partial_3 u_3)^2 \partial_2 u_2 dx := I_{391} + I_{392}.
\end{aligned}$$

因此

$$\begin{aligned}
|I_{391}| &= | -2 \int u_1 \partial_1 \partial_3 u_3 \partial_3 u_3 dx | \leq C \| \partial_1 \partial_3 u_3 \|_{L^2_{x_2 x_3} L^4_{x_1}} \times \\
&\| u_1 \|_{L^2_{x_1 x_2} L^\infty_{x_3}} \| \partial_3 u_3 \|_{L^2_{x_3} L^\infty_{x_2} L^4_{x_1}} \leq C \| \Lambda_1^{\frac{5}{4}} \partial_3 u_3 \|_{L^2} \times \\
&\| u_1 \|_{L^2}^{\frac{3}{5}} \| \Lambda_3^{\frac{5}{4}} u_1 \|_{L^2}^{\frac{2}{5}} \| \partial_3 u_3 \|_{L^2}^{\frac{2}{5}} \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_3 \|_{L^2}^{\frac{3}{5}} \leq \\
&\frac{\nu}{2048} \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_3 \|_{L^2}^2 + \\
&C \| u_1 \|_{L^2}^3 \| \Lambda_3^{\frac{5}{4}} u_1 \|_{L^2}^2 \| \partial_3 u_3 \|_{L^2}^2.
\end{aligned}$$

利用散度条件 $\nabla \cdot u = 0$ 可以得到

$$\begin{aligned}
I_{392} &= \int (\partial_3 u_3)^2 \partial_2 u_2 dx = \\
&\int (-\partial_1 u_1 - \partial_2 u_2) \partial_3 u_3 \partial_2 u_2 dx = \\
&-\int \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 dx - \int (\partial_2 u_2)^2 \partial_3 u_3 dx := \\
&I_{3921} + I_{3922}.
\end{aligned}$$

由分部积分可得

$$\begin{aligned}
I_{3921} &= -\int \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 dx = \int u_3 \partial_1 \partial_3 u_1 \partial_2 u_2 dx + \\
&\int u_3 \partial_1 u_1 \partial_2 \partial_3 u_2 dx := I_a + I_b,
\end{aligned}$$

因此

$$\begin{aligned}
I_a &= \int u_3 \partial_2 u_2 \partial_1 \partial_3 u_1 dx \leq \| \partial_1 \partial_3 u_1 \|_{L^2_{x_2 x_3} L^4_{x_1}} \| u_3 \|_{L^2_{x_3} L^4_{x_1 x_2}} \times \\
&\| \partial_2 u_2 \|_{L^2_{x_1} L^\infty_{x_3} L^4_{x_2}} \leq C \| \Lambda_1^{\frac{5}{4}} \partial_3 u_1 \|_{L^2} \| u_3 \|_{L^2}^{\frac{1}{2}} \times \\
&\| \nabla_h u_3 \|_{L^2}^{\frac{1}{2}} \| \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \leq \\
&\frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_3 u_1 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_3 u_2 \|_{L^2}^2 + \\
&C \| u_3 \|_{L^2}^2 \| \nabla_h u_3 \|_{L^2}^2 \| \Lambda_2^{\frac{5}{4}} u_2 \|_{L^2}^2,
\end{aligned}$$

同理得

$$\begin{aligned}
I_b &= \int u_3 \partial_2 \partial_3 u_2 \partial_1 u_1 dx \leq \| \partial_2 \partial_3 u_2 \|_{L^2_{x_1 x_3} L^4_{x_2}} \| u_3 \|_{L^2_{x_3} L^4_{x_1 x_2}} \times \\
&\| \partial_1 u_1 \|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} \leq C \| \Lambda_2^{\frac{5}{4}} \partial_3 u_2 \|_{L^2} \| u_3 \|_{L^2}^{\frac{1}{2}} \| \nabla_h u_3 \|_{L^2}^{\frac{1}{2}} \times \\
&\| \Lambda_1^{\frac{1}{4}} \partial_1 u_1 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_1^{\frac{1}{4}} \partial_1 u_1 \|_{L^2}^{\frac{1}{2}} \leq \\
&\frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_3 u_2 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_1^{\frac{5}{4}} \partial_3 u_1 \|_{L^2}^2 + \\
&C \| u_2 \|_{L^2}^2 \| \nabla_h u_2 \|_{L^2}^2 \| \Lambda_1^{\frac{5}{4}} u_1 \|_{L^2}^2.
\end{aligned}$$

根据 I_{342} 的估计, 可以得到

$$\begin{aligned}
I_{3922} &= \int u_3 \partial_2 \partial_3 u_3 \partial_2 u_2 dx \leq \| \partial_2 \partial_3 u_3 \|_{L^2_{x_2 x_3} L^4_{x_1}} \times \\
&\| u_3 \|_{L^2_{x_3} L^4_{x_1 x_2}} \| \partial_2 u_2 \|_{L^2_{x_1} L^\infty_{x_3} L^4_{x_2}} \leq C \| \Lambda_1^{\frac{1}{4}} \partial_2 \partial_3 u_3 \|_{L^2} \times \\
&\| u_3 \|_{L^2}^{\frac{1}{2}} \| \nabla_h u_3 \|_{L^2}^{\frac{1}{2}} \| \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \| \Lambda_3 \Lambda_2^{\frac{1}{4}} \partial_2 u_2 \|_{L^2}^{\frac{1}{2}} \leq \\
&\frac{\nu}{2048} \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_3 \|_{L^2}^2 + \frac{\nu}{2048} \| \Lambda_2^{\frac{5}{4}} \partial_3 u_2 \|_{L^2}^2 + \\
&C \| u_3 \|_{L^2}^2 \| \nabla_h u_3 \|_{L^2}^2 \| \Lambda_2^{\frac{5}{4}} u_2 \|_{L^2}^2.
\end{aligned}$$

正如之前提到的, u_1 包含三个方向的扩散, 因此 I_1 的估计比 I_2 和 I_3 更简单, 故在此省略了相关计算. 将上述所有估计综合在一起得到

$$\begin{aligned}
\frac{d}{dt} \| \nabla u \|^2_{L^2} + \nu \| \nabla \Lambda^{\frac{5}{4}} u_1 \|_{L^2}^2 + \nu \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla u_2 \|_{L^2}^2 + \\
\nu \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla u_3 \|_{L^2}^2 \leq C \left(\| \Lambda^{\frac{5}{4}} u_1 \|_{L^2}^2 + \right. \\
\left. \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_2 \|_{L^2}^2 + \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_3 \|_{L^2}^2 \right) (\| u \|^2_{L^2} + \\
\| u \|^3_{L^2}) \| \nabla u \|^2_{L^2}.
\end{aligned}$$

通过 Grönwall 不等式可知

$$\begin{aligned}
\| \nabla u \|^2_{L^2} + \nu \int_0^t \| \nabla \Lambda^{\frac{5}{4}} u_1 \|_{L^2}^2 d\tau + \\
\nu \int_0^t \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla u_2 \|_{L^2}^2 d\tau + \\
\nu \int_0^t \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla u_3 \|_{L^2}^2 d\tau \leq \| \nabla u_0 \|_{L^2}^2 \times \\
e^{\int_0^t \left(\| \Lambda^{\frac{5}{4}} u_1 \|_{L^2}^2 + \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_2 \|_{L^2}^2 + \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_3 \|_{L^2}^2 \right) ds} (\| u_0 \|_{L^2}^2 + \| u_0 \|^3_{L^2}) \leq \\
\| \nabla u_0 \|_{L^2}^2 \times \\
e^{C(\| u_0 \|_{L^2}^2 + \| u_0 \|^3_{L^2}) \int_0^t \left(\| \Lambda^{\frac{5}{4}} u_1 \|_{L^2}^2 + \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_2 \|_{L^2}^2 + \| (\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_3 \|_{L^2}^2 \right) ds} \leq
\end{aligned}$$

$$\begin{aligned} \|\nabla \mathbf{u}_0\|_{L^2}^2 e^{C(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{u}_0\|_{L^2}^4)} &\leq \\ \|\nabla \mathbf{u}_0\|_{L^2}^2 e^{C\|\mathbf{u}_0\|_{L^2}^4(1+\|\mathbf{u}_0\|_{L^2}^2)}, \end{aligned}$$

则全局存在性和正则性成立.

3 唯一性的证明

本节证明定理 2 中的唯一性结果.

设 $\tilde{\mathbf{u}}$ 表示解 $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ 的差, 即

$$\tilde{\mathbf{u}} := \mathbf{u}^{(1)} - \mathbf{u}^{(2)},$$

则其满足

$$\begin{cases} \partial_t \tilde{\mathbf{u}}_1 + \mathbf{u}^{(1)} \cdot \nabla \tilde{\mathbf{u}}_1 + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_1^{(2)} = -\partial_1 \tilde{p} - \nu \Lambda^{\frac{5}{2}} \tilde{\mathbf{u}}_1, \\ \partial_t \tilde{\mathbf{u}}_2 + \mathbf{u}^{(1)} \cdot \nabla \tilde{\mathbf{u}}_2 + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_2^{(2)} = -\partial_2 \tilde{p} - \\ \quad \nu \Lambda^{\frac{5}{2}} \tilde{\mathbf{u}}_2 - \nu \Lambda^{\frac{5}{2}} \tilde{\mathbf{u}}_2, \\ \partial_t \tilde{\mathbf{u}}_3 + \mathbf{u}^{(1)} \cdot \nabla \tilde{\mathbf{u}}_3 + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_3^{(2)} = -\partial_3 \tilde{p} - \\ \quad \nu \Lambda^{\frac{5}{2}} \tilde{\mathbf{u}}_3 - \nu \Lambda^{\frac{5}{2}} \tilde{\mathbf{u}}_3, \\ \nabla \cdot \tilde{\mathbf{u}} = 0, \\ \tilde{\mathbf{u}}(x, 0) = 0, \end{cases} \quad (15)$$

其中 $\tilde{p} = p^{(1)} - p^{(2)}$ 表示相应压力之差. 取方程(15)

与 $\tilde{\mathbf{u}}$ 的内积, 然后分部积分得

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{\mathbf{u}}\|_{L^2}^2 + \nu \|\Lambda^{\frac{5}{4}} \tilde{\mathbf{u}}_1\|_{L^2}^2 + \nu \|\Lambda^{\frac{5}{4}} \tilde{\mathbf{u}}_2\|_{L^2}^2 + \\ \nu \|\Lambda^{\frac{5}{4}} \tilde{\mathbf{u}}_3\|_{L^2}^2 = J, \end{aligned}$$

其中

$$\begin{aligned} J = \int (\tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)}) \cdot \tilde{\mathbf{u}} dx = - \int (\tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_1^{(2)}) \cdot \tilde{\mathbf{u}}_1 dx - \\ \int (\tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_2^{(2)}) \cdot \tilde{\mathbf{u}}_2 dx - \int (\tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_3^{(2)}) \cdot \tilde{\mathbf{u}}_3 dx := \\ J_1 + J_2 + J_3. \end{aligned}$$

因为对 $\mathbf{u}_1^{(2)}$ 和 $\tilde{\mathbf{u}}_1$ 有更多的控制, 所以项 J_1 的估计比 J_2 和 J_3 更容易. 这里只计算 J_2 的估计, J_3 中的项也可以类似地估计.

$$\begin{aligned} J_2 = - \int \tilde{\mathbf{u}}_1 \partial_1 \mathbf{u}_2^{(2)} \tilde{\mathbf{u}}_2 dx - \int \tilde{\mathbf{u}}_2 \partial_2 \mathbf{u}_2^{(2)} \tilde{\mathbf{u}}_2 dx - \\ \int \tilde{\mathbf{u}}_3 \partial_3 \mathbf{u}_2^{(2)} \tilde{\mathbf{u}}_2 dx := J_{21} + J_{22} + J_{23}. \end{aligned}$$

由 Hölder 不等式和引理 1 得

$$\begin{aligned} |J_{21}| &\leq \|\partial_1 \mathbf{u}_2^{(2)}\|_{L^2} \|\tilde{\mathbf{u}}_1\|_{L_{x_2}^2 L_{x_3}^{\infty} L_{x_1}^4} \|\tilde{\mathbf{u}}_2\|_{L_{x_1}^2 L_{x_2}^{\infty} L_{x_3}^4} \leq \\ &C \|\partial_1 \mathbf{u}_2^{(2)}\|_{L^2} \|\Lambda_1^{\frac{1}{4}} \tilde{\mathbf{u}}_1\|_{L_{x_2}^2 L_{x_3}^{\infty} L_{x_1}^4} \|\Lambda_1^{\frac{1}{4}} \tilde{\mathbf{u}}_2\|_{L_{x_1}^2 L_{x_2}^{\infty} L_{x_3}^4} \leq \\ &C \|\partial_1 \mathbf{u}_2^{(2)}\|_{L^2} \|\Lambda_1^{\frac{1}{4}} \tilde{\mathbf{u}}_1\|_{L^2}^{\frac{1}{2}} \|\Lambda_1^{\frac{1}{4}} \Lambda_3 \tilde{\mathbf{u}}_1\|_{L^2}^{\frac{1}{2}} \|\Lambda_1^{\frac{1}{4}} \tilde{\mathbf{u}}_2\|_{L^2}^{\frac{1}{2}} \times \\ &\|\Lambda_1^{1/4} \Lambda_2 \tilde{\mathbf{u}}_2\|_{L^2}^{1/2}, \end{aligned}$$

其中使用了插值不等式

$$\begin{aligned} \|\Lambda_1^{\frac{1}{4}} \tilde{\mathbf{u}}_1\|_{L^2} &\leq C \|\tilde{\mathbf{u}}_1\|_{L^2}^{\frac{4}{5}} \|\Lambda_1^{\frac{5}{4}} \tilde{\mathbf{u}}_1\|_{L^2}^{\frac{1}{5}}, \\ \|\Lambda_1^{\frac{1}{4}} \tilde{\mathbf{u}}_2\|_{L^2} &\leq C \|\tilde{\mathbf{u}}_2\|_{L^2}^{\frac{4}{5}} \|\Lambda_1^{\frac{5}{4}} \tilde{\mathbf{u}}_2\|_{L^2}^{\frac{1}{5}}, \end{aligned}$$

从而

$$\begin{aligned} |J_{21}| &\leq C \|\partial_1 \mathbf{u}_2^{(2)}\|_{L^2} \|\tilde{\mathbf{u}}_1\|_{L^2}^{\frac{2}{5}} \|\tilde{\mathbf{u}}_2\|_{L^2}^{\frac{2}{5}} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \tilde{\mathbf{u}}_1\|_{L^2}^{\frac{3}{5}} \times \\ &\|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \tilde{\mathbf{u}}_2\|_{L^2}^{\frac{3}{5}} \leq \frac{\nu}{128} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \tilde{\mathbf{u}}_1\|_{L^2}^2 + \\ &\frac{\nu}{128} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \tilde{\mathbf{u}}_2\|_{L^2}^2 + C \|\partial_1 \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{5}{2}} \|\tilde{\mathbf{u}}_1\|_{L^2} \|\tilde{\mathbf{u}}_2\|_{L^2}. \end{aligned}$$

类似地,

$$\begin{aligned} |J_{22}| &\leq \|\tilde{\mathbf{u}}_2\|_{L^2} \|\tilde{\mathbf{u}}_2\|_{L_{x_3}^2 L_{x_1}^{\infty} L_{x_2}^4} \|\partial_2 \mathbf{u}_2^{(2)}\|_{L_{x_1}^2 L_{x_3}^{\infty} L_{x_2}^4} \leq \\ &C \|\tilde{\mathbf{u}}_2\|_{L^2} \|\Lambda_2^{\frac{1}{4}} \tilde{\mathbf{u}}_2\|_{L_{x_3}^2 L_{x_1}^{\infty} L_{x_2}^4} \|\Lambda_2^{\frac{1}{4}} \partial_2 \mathbf{u}_2^{(2)}\|_{L_{x_1}^2 L_{x_3}^{\infty} L_{x_2}^4} \leq \\ &C \|\tilde{\mathbf{u}}_2\|_{L^2} \|\Lambda_2^{\frac{1}{4}} \tilde{\mathbf{u}}_2\|_{L^2}^{\frac{1}{2}} \|\Lambda_2^{\frac{1}{4}} \Lambda_1 \tilde{\mathbf{u}}_2\|_{L^2}^{\frac{1}{2}} \|\Lambda_2^{\frac{1}{4}} \partial_2 \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{1}{2}} \times \\ &\|\Lambda_2^{\frac{1}{4}} \Lambda_3 \partial_2 \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{1}{2}} \leq C \|\tilde{\mathbf{u}}_2\|_{L^2}^{\frac{7}{5}} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \tilde{\mathbf{u}}_2\|_{L^2}^{\frac{3}{5}} \times \\ &\|\Lambda_2^{\frac{5}{4}} \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{1}{2}} \|\Lambda_2^{\frac{5}{4}} \partial_3 \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{1}{2}} \leq \\ &\frac{\nu}{128} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \tilde{\mathbf{u}}_2\|_{L^2}^2 + \\ &C \|\tilde{\mathbf{u}}_2\|_{L^2}^2 \|\Lambda_2^{\frac{5}{4}} \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{5}{2}} \|\Lambda_2^{\frac{5}{4}} \partial_3 \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{5}{2}}. \end{aligned}$$

对 J_{23} 的估计与 J_{22} 不同. 注意到在 J_{23} 中只有 $\mathbf{u}_2, \mathbf{u}_3$ 和 $\partial_3 \mathbf{u}_2^{(2)}$, 因此有

$$\begin{aligned} |J_{23}| &\leq \|\tilde{\mathbf{u}}_3\|_{L^2} \|\tilde{\mathbf{u}}_2\|_{L_{x_3}^2 L_{x_1}^{\infty} L_{x_2}^4} \|\partial_3 \mathbf{u}_2^{(2)}\|_{L_{x_1}^2 L_{x_3}^{\infty} L_{x_2}^4} \leq \\ &C \|\tilde{\mathbf{u}}_3\|_{L^2} \|\tilde{\mathbf{u}}_2\|_{L^2}^{\frac{2}{5}} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \tilde{\mathbf{u}}_2\|_{L^2}^{\frac{3}{5}} \|\partial_3 \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{2}{5}} \times \\ &\|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \partial_3 \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{3}{5}} \leq \frac{\nu}{128} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \tilde{\mathbf{u}}_2\|_{L^2}^2 + \\ &C \|\tilde{\mathbf{u}}_3\|_{L^2}^{\frac{10}{7}} \|\tilde{\mathbf{u}}_2\|_{L^2}^{\frac{4}{7}} \|\partial_3 \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{4}{7}} \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \partial_3 \mathbf{u}_2^{(2)}\|_{L^2}^{\frac{6}{7}}. \end{aligned}$$

结合 J_1, J_2 和 J_3 的估计得

$$\begin{aligned} \frac{d}{dt} \|\tilde{\mathbf{u}}\|_{L^2}^2 + \nu \|\Lambda^{\frac{5}{4}} \tilde{\mathbf{u}}_1\|_{L^2}^2 + \nu \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \tilde{\mathbf{u}}_2\|_{L^2}^2 + \\ \nu \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \tilde{\mathbf{u}}_3\|_{L^2}^2 \leq C \|\nabla \mathbf{u}^{(2)}\|_{L^2}^{\frac{5}{2}} \|\tilde{\mathbf{u}}\|_{L^2}^2 + \\ C \|\tilde{\mathbf{u}}\|_{L^2}^2 \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \mathbf{u}^{(2)}\|_{L^2}^{\frac{5}{2}} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \nabla \mathbf{u}^{(2)}\|_{L^2}^{\frac{5}{2}} + \\ C \|\tilde{\mathbf{u}}\|_{L^2}^2 \|\partial_3 \mathbf{u}^{(2)}\|_{L^2}^{\frac{4}{2}} \|(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \nabla \mathbf{u}^{(2)}\|_{L^2}^{\frac{6}{2}}. \end{aligned}$$

利用 Grönwall 不等式可知, 如果 $\mathbf{u}_0^{(1)} = \mathbf{u}_0^{(2)}$, 则 $\mathbf{u}^{(1)} \equiv \mathbf{u}^{(2)}$, 即定理 2 中的唯一性成立.

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Well-posedness of three dimensional Navier-Stokes equations with partial fractional diffusion terms

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Abstract: It is a well-known fact that the Navier-Stokes equation with hyperdissipation $(-\Delta)^{5/4}$ is well-posed. There are many scholars committing themselves to solving the problem with weaker hyperdissipation. When some components of hyperdissipation in different spatial directions are removed from the Navier-Stokes equations, the existence as well as uniqueness under some extra conditions is thus proved. In this paper, we derive the existence and uniqueness of the problem with hyperdissipation in x_3 direction removed from the equations of u_2 and u_3 . It is worth mentioning that the well-posedness result shall not be obtained by the present method if the hyperdissipation in x_3 direction is removed from all three equations.

Key words: Navier-Stokes equations; partial diffusion; fractional diffusion; hyperdissipation; well-posedness

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