

# Möbius 几何中的多分量 Camassa-Holm 方程

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**摘要:** 研究了一般 Möbius 几何中的曲线流, 证明了一类多分量的 Camassa-Holm 方程等价于 Möbius 几何中的一个不变曲线流, 此方程是两分量 Camassa-Holm 方程的多分量推广, 也可以看成是一类多分量 KdV 系统的对偶可积系统. 最后得到了此方程的一个退化情形的尖峰孤子解.

**关键词:** 不变曲线流; Möbius 几何; Camassa-Holm 方程; 多分量 Camassa-Holm 系统; 尖峰孤子解

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1993 年, Camassa 等<sup>[1]</sup>在浅水波状态下欧拉方程的哈密顿量中使用渐近展开获得 Camassa-Holm (CH) 方程. 由于几个不平凡的特征, CH 方程的各类多分量可积方程在过去二十多年得到了广泛发展. 一方面, 一些 CH 型方程被证明描述了具有不变度量的测地线流<sup>[2-3]</sup>或来自某些几何曲线流<sup>[4-5]</sup>; 另一方面, 这些 CH 型方程是非线性色散波方程, 并且大多数都具有非光滑的孤子结构, 例如单峰和多峰孤子<sup>[1,6]</sup>. 2003 年, Sanders 等<sup>[7]</sup>得到嵌入在具有恒定曲率的  $n$  维黎曼流形中曲线联络<sup>[8]</sup>演变的结构方程, 可导出辛算子、哈密顿算子和遗传算子. 这就提供了有限维几何、无限维几何和可积系统之间的自然联系. 这些结果可以通过使用自然或平行标架获得. 2006 年, Sanders 等<sup>[9]</sup>把以上结果推广到共形几何上, 得到了嵌入在具有恒定曲率的  $n$  维共形流形中曲线平行标架流动的结构方程, 这将得到两个相容的哈密顿算子. Sanders 等利用共形几何中 Möbius 模型的 Cartan 公式规则<sup>[8]</sup>得到一个结构方程. 同年, Chen 等<sup>[10]</sup>对文献[1]中的 Lax 对进行推广, 得到了两分量的 CH 方程. 2013 年, Qu 等<sup>[11]</sup>证明了一些多分量可积系统可通过三维欧几里得几何中的不变曲线流获得. 2021 年, Kang 等<sup>[12]</sup>在 Sanders 等<sup>[9]</sup>的工作基础上把 Möbius 几何的李代数形式推广到了更一般的形式, 从而得到了一个

$(1+n)$  分量的 KdV 方程和一个  $(1+n)$  分量的 Camassa-Holm 方程.

本文在文献[12]的基础上把 Möbius 几何的李代数形式进一步推广, 证明一个  $(1+n+r)$  分量的 KdV 方程和  $(1+n+r)$  分量的 Camassa-Holm 方程可以从这一般 Möbius 几何中的曲线流中得到. 利用伸缩变换进一步研究这个多分量 Camassa-Holm 方程的柯西问题, 最终得到一个退化情形的尖峰孤子解.

## 1 一般 Möbius 几何中的可积方程

2006 年, Sanders 等<sup>[9]</sup>研究了以下李代数确定的 Möbius 几何中的曲线流,

$$\begin{pmatrix} -\epsilon & \mathbf{v}^T & 0 \\ \boldsymbol{\theta} & \boldsymbol{\xi} & \mathbf{v} \\ 0 & \boldsymbol{\theta}^T & \epsilon \end{pmatrix}, \quad (1)$$

式中:  $\epsilon \in \mathbb{R}$ ,  $\mathbf{v} = (v_1, v_2, \dots, v_n)^T \in \mathbb{R}^n$ ;  $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n)^T \in \mathbb{R}^n$  以及  $\boldsymbol{\xi} = (\xi_{ij})_{n \times n} \in \mathfrak{o}(n)$ . 从该几何的曲线流, Sanders 等推导出了相应的多分量 KdV 型可积方程

$$\begin{cases} \psi_t = \psi_{xxx} + \frac{3}{2}(\psi^2 - \langle \mathbf{k}, \mathbf{k} \rangle - \langle \mathbf{u}, \mathbf{u} \rangle)_x, \\ \mathbf{k}_t = \mathbf{k}_{xxx} + 3(\psi \mathbf{k})_x, \\ \mathbf{u}_t = \mathbf{u}_{xxx} + 3\psi \mathbf{u}_x, \end{cases} \quad (2)$$

式中:  $\psi(t, x)$  是一个标量函数;  $\mathbf{k}(t, x) = (k_2(t, x), \dots, k_m(t, x))^T$ ;  $\mathbf{u}(t, x) = (u_{m+1}(t, x), \dots, u_n(t, x))^T$ ;  $1 \leq m \leq n$ . 式(1)中元素和式(2)中变量的联系如下:

$$\begin{aligned} \epsilon &= \sigma_x, \quad \theta_1 = \psi = -\sigma + \frac{1}{2} \sum_{l=m+1}^n u_l^2, \\ \theta_i &= k_i, \quad 2 \leq i \leq m, \\ \theta_j &= -u_{j,x}, \quad \xi_{il} = u_{i,x}, \quad m+1 \leq i, j \leq n, \\ \xi_{ij} &= \begin{cases} 0, & 2 \leq j \leq m, \\ -k_i u_j, & 1 \leq i \leq m < j \leq n, \\ u_j u_{i,x} - u_i u_{j,x}, & m+1 \leq i < j \leq n, \end{cases} \\ v_i &= \begin{cases} \sigma_{xx} + \psi \sigma + \sum_{j=2}^m k_j^2, & i=1, \\ k_{i,xx} + 3\psi k_i + \sigma k_i, & 2 \leq i \leq m, \\ \sigma_x u_i - \sigma u_{i,x}, & m+1 \leq i \leq n. \end{cases} \end{aligned}$$

文献[12]从 Sanders 等<sup>[9]</sup>的工作出发, 提出研究由更一般的李代数

$$\begin{pmatrix} -\epsilon & \mathbf{w}^T & \mathbf{u}^T & 0 \\ \theta_1 & \alpha & \xi & \mathbf{w} \\ \theta_2 & \xi^T & \beta & -\mathbf{u} \\ 0 & \theta_1^T & -\theta_2^T & \epsilon \end{pmatrix}$$

生成的 Möbius 几何中的几何曲线流, 其中  $\epsilon \in \mathbb{R}$ ,  $\theta_1 \in \mathbb{R}^l$ ,  $\theta_2 \in \mathbb{R}^n$ ,  $\mathbf{w} \in \mathbb{R}^l$ ,  $\mathbf{u} \in \mathbb{R}^n$ ,  $\alpha \in \mathfrak{o}(l)$ ,  $\beta \in \mathfrak{o}(n)$ , 且  $\xi$  是一个  $l \times n$  阶矩阵. 由此推导出了相应的多分量 KdV 型可积方程:

$$\begin{cases} w_t = w_{xxx} + \frac{3}{2}(w^2 + \langle \mathbf{u}, \mathbf{u} \rangle)_x, \\ \mathbf{u}_t = \mathbf{u}_{xxx} + 3(\mathbf{w}\mathbf{u})_x, \end{cases}$$

式中:  $w(t, x)$  是一个标量函数;  $\mathbf{u}(t, x)$  是一个  $n$  维向量函数. 受文献[9,12]等工作的启发, 本文研究李代数

$$\begin{pmatrix} -\epsilon & \mathbf{w}^T & \mathbf{u}^T & \mathbf{v}^T & 0 \\ \theta_1 & \alpha & \xi_3 & \xi_1 & \mathbf{w} \\ \theta_2 & \xi_3^T & \beta & \xi_2 & -\mathbf{u} \\ \theta_3 & -\xi_1^T & \xi_2^T & \gamma & \mathbf{v} \\ 0 & \theta_1^T & -\theta_2^T & \theta_3^T & \epsilon \end{pmatrix}$$

确定的 Möbius 几何中的曲线流. 下面将证明一个  $(1+n+r)$  分量的 KdV 方程和  $(1+n+r)$  分量的 Camassa-Holm 方程可以从这一般 Möbius 几何中的曲线流中得到.

本节将得到一些由 Möbius 几何  $\text{Möb}(n+l, \mathbb{R})$  曲线演化产生的多分量方程. Möbius 几何是一类特殊的 Klein 几何, 其主群是维数为  $(n+l+2)$  的射影 Lorentz 群. 关于 Möbius 几何更多的细节请参考文献[9,12-13].

设  $I_i$  为  $i \times i$  阶矩阵,  $\mathbf{0}_{i \times j}$  为零矩阵,  $\mathbf{0}_i$  为  $i$  元零向量.

引入 Lorentz 群

$$L_{n+l+1,1}(\mathbb{R}) = \{g \in GL(n+l+2, \mathbb{R}) | g^T \Xi_{n+l+1,1} g = \Xi_{n+l+1,1}\}$$

和 Möbius 群

$$\text{Möb}(n+l, \mathbb{R}) = L_{n+l+1,1}(\mathbb{R}) / \{\pm I\},$$

$$\text{其中 } \Xi_{n+l+1,1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_{n+l} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

设

$$\mathcal{H} = \{h \in \text{Möb}(n+l, \mathbb{R}) | h[\mathbf{e}_0] = [\mathbf{e}_0]\},$$

式中:  $[\mathbf{e}_0]$  表示  $\mathbf{e}_0$  在  $P(\mathbb{R}^{n+l+2})$  中的类;  $P(\mathbb{R}^{n+l+2})$  是  $\mathbb{R}^{n+l+2}$  上的投影空间. 群  $\text{Möb}(n+l, \mathbb{R})$  和  $\mathcal{H}$  相应的李代数用  $\mathfrak{g}$  和  $\mathfrak{h}$  表示, 形式为

$$\begin{pmatrix} -\epsilon & \mathbf{w}^T & \mathbf{u}^T & \mathbf{v}^T & 0 \\ \theta_1 & \alpha & \xi_3 & \xi_1 & \mathbf{w} \\ \theta_2 & \xi_3^T & \beta & \xi_2 & -\mathbf{u} \\ \theta_3 & -\xi_1^T & \xi_2^T & \gamma & \mathbf{v} \\ 0 & \theta_1^T & -\theta_2^T & \theta_3^T & \epsilon \end{pmatrix} \in \mathfrak{g},$$

$$\begin{pmatrix} -\epsilon & \mathbf{w}^T & \mathbf{u}^T & \mathbf{v}^T & 0 \\ 0 & \alpha & \xi_3 & \xi_1 & \mathbf{w} \\ 0 & 0 & \beta & \xi_2 & -\mathbf{u} \\ 0 & 0 & 0 & \gamma & \mathbf{v} \\ 0 & 0 & 0 & 0 & \epsilon \end{pmatrix} \in \mathfrak{h},$$

式中:  $\epsilon \in \mathbb{R}$ ,  $\theta_1 \in \mathbb{R}^m$ ,  $\theta_2 \in \mathbb{R}^n$ ,  $\theta_3 \in \mathbb{R}^r$ ,  $\mathbf{w} \in \mathbb{R}^m$ ,  $\mathbf{u} \in \mathbb{R}^n$ ,  $\mathbf{v} \in \mathbb{R}^r$ ,  $\alpha \in \mathfrak{o}(m)$ ,  $\beta \in \mathfrak{o}(n)$ ,  $\gamma \in \mathfrak{o}(r)$ ;  $\xi_1$  是  $m \times r$  阶矩阵,  $\xi_2$  是  $n \times r$  阶矩阵,  $\xi_3$  是  $m \times n$  阶矩阵.

给定 Möbius 几何  $\text{Möb}(n+l, \mathbb{R})$  上的一条曲线  $\gamma(t, x)$ , 设  $\mathbf{D}_x$  为曲线  $\gamma(t, x)$  在点  $p$  上的切向量. 移动标架  $\mathbf{e}$  是切空间的基, 其分量为  $e_i, i=1, \dots, n+l$ . 这里一个自然的问题是确定演化向量  $\mathbf{D}_t$ , 使得它满足当  $t=0$  时  $\mathbf{D}_t = \mathbf{D}_x$ , 以及当任意  $t>0$  时  $[\mathbf{D}_x, \mathbf{D}_t] = 0$ . 使用  $\omega$  来表示 Cartan 矩阵, 形式为

$$\omega(D_x) = \begin{pmatrix} -\epsilon & \mathbf{w}^T & \mathbf{u}^T & \mathbf{v}^T & 0 \\ \theta_1 & \alpha & \xi_3 & \xi_1 & \mathbf{w} \\ \theta_2 & \xi_3^T & \beta & \xi_2 & -\mathbf{u} \\ \theta_3 & -\xi_1^T & \xi_2^T & \gamma & \mathbf{v} \\ 0 & \theta_1^T & -\theta_2^T & \theta_3^T & \epsilon \end{pmatrix},$$

$$\omega(D_t) = \begin{pmatrix} -\tilde{\epsilon} & \tilde{\mathbf{w}}^T & \tilde{\mathbf{u}}^T & \tilde{\mathbf{v}}^T & 0 \\ \tilde{\theta}_1 & \tilde{\alpha} & \tilde{\xi}_3 & \tilde{\xi}_1 & \tilde{\mathbf{w}} \\ \tilde{\theta}_2 & \tilde{\xi}_3^T & \tilde{\beta} & \tilde{\xi}_2 & -\tilde{\mathbf{u}} \\ \tilde{\theta}_3 & -\tilde{\xi}_1^T & \tilde{\xi}_2^T & \tilde{\gamma} & \tilde{\mathbf{v}} \\ 0 & \tilde{\theta}_1^T & -\tilde{\theta}_2^T & \tilde{\theta}_3^T & \tilde{\epsilon} \end{pmatrix}, \quad (3)$$

式中:  $\theta_1$ 、 $\theta_2$  和  $\theta_3$  为常向量;  $\mathbf{w}$ 、 $\mathbf{u}$ 、 $\mathbf{v}$ 、 $\alpha$ 、 $\beta$ 、 $\gamma$ 、 $\xi$  依赖于曲线  $\gamma$  的曲率. 特别地, 对于给定的 Cartan 矩阵  $\omega$ , 其中联络  $d$  可以由  $de = \omega e$  确定. 由结构方程  $d\omega - \omega \wedge \omega = 0$ , 有

$$D_t \omega(D_x) - D_x \omega(D_t) + [\omega(D_x), \omega(D_t)] = 0. \quad (4)$$

将式(3)代入式(4), 得到以下方程:

$$\begin{aligned} \tilde{\epsilon}_x + \mathbf{w}^T \tilde{\theta}_1 - \tilde{\mathbf{w}}^T \theta_1 + \mathbf{u}^T \tilde{\theta}_2 - \tilde{\mathbf{u}}^T \theta_2 + \mathbf{v}^T \tilde{\theta}_3 - \tilde{\mathbf{v}}^T \theta_3 &= 0, \\ \mathbf{w}_t^T - \tilde{\mathbf{w}}_x^T - \epsilon \tilde{\mathbf{w}}^T + \mathbf{w}^T \tilde{\alpha} + \mathbf{u}^T \tilde{\xi}_3^T - \mathbf{v}^T \tilde{\xi}_1^T + \tilde{\epsilon} \mathbf{w}^T - \tilde{\mathbf{w}}^T \alpha - \tilde{\mathbf{u}}^T \xi_3^T + \tilde{\mathbf{v}}^T \xi_1^T &= 0, \\ \mathbf{u}_t^T - \tilde{\mathbf{u}}_x^T - \epsilon \tilde{\mathbf{u}}^T + \mathbf{w}^T \tilde{\xi}_3 + \mathbf{u}^T \tilde{\beta} + \mathbf{v}^T \tilde{\xi}_2^T + \tilde{\epsilon} \mathbf{u}^T - \tilde{\mathbf{w}}^T \xi_3 - \tilde{\mathbf{u}}^T \beta - \tilde{\mathbf{v}}^T \xi_2^T &= 0, \\ \mathbf{v}_t^T - \tilde{\mathbf{v}}_x^T - \epsilon \tilde{\mathbf{v}}^T + \mathbf{w}^T \tilde{\xi}_1 + \mathbf{u}^T \tilde{\xi}_2 + \mathbf{v}^T \tilde{\gamma} + \tilde{\epsilon} \mathbf{v}^T - \tilde{\mathbf{w}}^T \xi_1 - \tilde{\mathbf{u}}^T \xi_2 - \tilde{\mathbf{v}}^T \gamma &= 0, \\ \mathbf{w}^T \tilde{\mathbf{w}} - \mathbf{u}^T \tilde{\mathbf{u}} + \mathbf{v}^T \tilde{\mathbf{v}} - \tilde{\mathbf{w}}^T \mathbf{w} + \tilde{\mathbf{u}}^T \mathbf{u} - \tilde{\mathbf{v}}^T \mathbf{v} &= 0, \\ \theta_{1,t} - \tilde{\theta}_{1,x} - \tilde{\epsilon} \theta_1 + \alpha \tilde{\theta}_1 + \xi_3 \tilde{\theta}_2 + \xi_1 \tilde{\theta}_3 + \epsilon \tilde{\theta}_1 - \tilde{\alpha} \theta_1 - \tilde{\xi}_3 \theta_2 - \tilde{\xi}_1 \theta_3 &= 0, \\ \theta_{2,t} - \tilde{\theta}_{2,x} - \tilde{\epsilon} \theta_2 + \xi_3^T \tilde{\theta}_1 + \beta \tilde{\theta}_2 + \xi_2 \tilde{\theta}_3 + \epsilon \tilde{\theta}_2 - \tilde{\xi}_3^T \theta_1 - \tilde{\beta} \theta_2 - \tilde{\xi}_2 \theta_3 &= 0, \\ \theta_{3,t} - \tilde{\theta}_{3,x} - \tilde{\epsilon} \theta_3 + \xi_1^T \tilde{\theta}_1 + \xi_2^T \tilde{\theta}_2 + \gamma \tilde{\theta}_3 + \epsilon \tilde{\theta}_3 + \tilde{\xi}_1^T \theta_1 - \tilde{\xi}_2^T \theta_2 - \tilde{\gamma} \theta_3 &= 0, \\ \alpha_t - \tilde{\alpha}_x + \theta_1 \tilde{\mathbf{w}}^T + \xi_3 \tilde{\xi}_3^T - \xi_1 \tilde{\xi}_1^T + \mathbf{w} \tilde{\theta}_1^T - \tilde{\theta}_1 \mathbf{w}^T - \tilde{\xi}_3 \xi_3^T + \tilde{\xi}_1 \xi_1^T - \tilde{\mathbf{w}} \theta_1^T &= 0, \\ \beta_t - \tilde{\beta}_x + \theta_2 \tilde{\mathbf{u}}^T + \tilde{\xi}_3 \xi_3^T + \xi_2 \tilde{\xi}_2^T + \mathbf{u} \tilde{\theta}_2^T - \tilde{\theta}_2 \mathbf{u}^T - \tilde{\xi}_3^T \xi_3 - \tilde{\xi}_2^T \xi_2 - \tilde{\mathbf{u}} \theta_2^T &= 0, \end{aligned}$$

$$\begin{aligned} \gamma_t - \tilde{\gamma}_x + \theta_3 \tilde{\mathbf{v}}^T - \tilde{\xi}_1 \xi_1^T + \xi_2^T \tilde{\xi}_2 + \mathbf{v} \tilde{\theta}_3^T - \tilde{\theta}_3 \mathbf{v}^T + \tilde{\xi}_1^T \xi_1 - \tilde{\xi}_2^T \xi_2 - \tilde{\mathbf{v}} \theta_3^T &= 0, \\ \xi_{3,t} - \tilde{\xi}_{3,x} + \theta_1 \tilde{\mathbf{u}}^T + \alpha \tilde{\xi}_3 + \xi_3 \tilde{\beta} + \xi_1 \tilde{\xi}_2^T - \mathbf{w} \tilde{\theta}_2^T - \tilde{\theta}_1 \mathbf{u}^T - \tilde{\alpha} \xi_3 - \tilde{\xi}_3 \beta - \tilde{\xi}_1 \xi_2^T + \mathbf{w}^T \theta_2^T &= 0, \\ \xi_{1,t} - \tilde{\xi}_{1,x} + \theta_1 \tilde{\mathbf{v}}^T + \alpha \tilde{\xi}_1 + \xi_3 \tilde{\xi}_2 + \xi_1 \tilde{\gamma} + \mathbf{w} \tilde{\theta}_3^T - \tilde{\theta}_1 \mathbf{v}^T - \tilde{\alpha} \xi_1 - \tilde{\xi}_3 \xi_2 - \tilde{\xi}_1 \gamma - \tilde{\mathbf{w}} \theta_3^T &= 0, \\ \xi_{2,t} - \tilde{\xi}_{2,x} + \theta_2 \tilde{\mathbf{v}}^T + \xi_3^T \tilde{\xi}_1 + \beta \tilde{\xi}_2 + \xi_2 \tilde{\gamma} - \mathbf{u} \tilde{\theta}_3^T - \tilde{\theta}_2 \mathbf{v}^T - \tilde{\xi}_3^T \xi_1 - \tilde{\beta} \xi_2 - \tilde{\xi}_2 \gamma + \tilde{\mathbf{u}} \theta_3^T &= 0. \end{aligned} \quad (5)$$

给定  $\mathbf{w}$ 、 $\mathbf{u}$ 、 $\alpha$ 、 $\beta$ 、 $\gamma$  和  $\xi$ , 求解此系统也不容易. 为简单起见, 设  $l=2$ ,  $\theta_1=\theta_1=1$ ,  $\alpha=\tilde{\alpha}=0$ ,  $\beta=0$ ,  $\gamma=\mathbf{0}_{r \times r}$ ,  $\xi_1=\mathbf{0}_{m \times r}^T$ ,  $\xi_3=\mathbf{0}_{m \times n}^T$ ,  $\xi_2=\tilde{\xi}_2=\mathbf{0}_{n \times r}^T$ ,  $\theta_2=\mathbf{0}_n$ ,  $\theta_3=\mathbf{0}_r$ . 考虑两种特殊情况.

情形1 当  $\mathbf{w}=\tilde{\mathbf{w}}=-\mathbf{w}$ , 设  $\tilde{\gamma}=\mathbf{0}_{r \times r}$ ,  $\tilde{\beta}=\mathbf{0}_{n \times n}$ , 方程(5)化简为以下形式:

$$\begin{aligned} \tilde{\epsilon}_x - \mathbf{w} \tilde{\theta}_1 - \tilde{\mathbf{w}} + \mathbf{u}^T \tilde{\theta}_2 + \mathbf{v}^T \tilde{\theta}_3 &= 0, \\ \mathbf{w}_t + \tilde{\mathbf{w}}_x - \mathbf{u}^T \tilde{\xi}_3^T - \mathbf{v}^T \tilde{\xi}_1^T + \tilde{\epsilon} \mathbf{w} &= 0, \\ \mathbf{u}_t^T - \tilde{\mathbf{u}}_x^T - \mathbf{w} \tilde{\xi}_3 + \tilde{\epsilon} \mathbf{u}^T &= 0, \\ \mathbf{v}_t^T - \tilde{\mathbf{v}}_x^T - \mathbf{w} \tilde{\xi}_1 + \tilde{\epsilon} \mathbf{v}^T &= 0, \\ \tilde{\epsilon} + \tilde{\theta}_{1,x} &= 0, \\ \tilde{\theta}_{2,x} + \tilde{\xi}_3^T &= 0, \\ -\tilde{\theta}_{3,x} + \tilde{\xi}_1^T &= 0. \end{aligned}$$

显然, 方程有以下解:

$$\begin{aligned} \tilde{\mathbf{w}} &= -\mathbf{w}_{xx} - \mathbf{w}^2 - \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle, \\ \tilde{\mathbf{u}}_i &= \mathbf{u}_{i,xx} + 2\mathbf{w} \mathbf{u}_i, \quad i=1, 2, \dots, n, \\ \tilde{\mathbf{v}}_i &= \mathbf{v}_{i,xx} + 2\mathbf{w} \mathbf{v}_i, \quad i=1, 2, \dots, n, \\ \tilde{\epsilon} &= -\mathbf{w}_x, \\ \tilde{\theta}_1 &= \mathbf{w}, \\ \tilde{\xi}_1 &= \mathbf{v}^T, \\ \tilde{\xi}_3 &= \mathbf{u}^T, \\ \tilde{\theta}_2 &= -\mathbf{u}, \\ \tilde{\theta}_3 &= \mathbf{v}, \end{aligned}$$

从而标量函数  $w(t, x)$ 、向量函数  $\mathbf{u}(t, x)$  和  $\mathbf{v}(t, x)$  满足以下  $(1+n+r)$  分量 KdV 方程:

$$\begin{cases} w_t = w_{xxx} + \frac{3}{2}(w^2 + \langle \mathbf{u}, \mathbf{u} \rangle - \frac{1}{3}\langle \mathbf{v}, \mathbf{v} \rangle)_x, \\ \mathbf{u}_t = \mathbf{u}_{xxx} + 3(\mathbf{w} \mathbf{u})_x, \\ \mathbf{v}_t = \mathbf{v}_{xxx} + 3(\mathbf{w} \mathbf{v})_x. \end{cases} \quad (6)$$

情形 2 当  $w = \tilde{w} = -\rho + \frac{1}{2}$ ,  $\rho = w - w_{xx}$  并设  $u \mapsto m = u - u_{xx}$ ,  $v \mapsto n = v - v_{xx}$ , 方程(5)变为

$$\begin{aligned}\tilde{\beta}_x &= m\tilde{\theta}_2^T - \tilde{\theta}_2 m^T, \\ \tilde{\gamma}_x &= n\tilde{\theta}_3^T - \tilde{\theta}_3 n^T, \\ \tilde{\epsilon} + \tilde{\theta}_{1,x} &= 0, \\ \tilde{\theta}_{2,x} + \tilde{\xi}_3^T &= 0, \\ -\tilde{\theta}_{3,x} + \tilde{\xi}_1^T &= 0, \\ \tilde{u}^T - \tilde{\xi}_{3,x} + (\rho - \frac{1}{2})u^T - \tilde{\theta}_1 m^T &= 0, \\ \tilde{v}^T - \tilde{\xi}_{1,x} - (\rho - \frac{1}{2})v^T - \tilde{\theta}_1 n^T &= 0, \\ \tilde{w} - \tilde{\epsilon}_x + (\rho - \frac{1}{2})\tilde{\theta}_1 - m^T \tilde{\theta}_2 - n^T \tilde{\theta}_3 &= 0, \\ \rho_t + \tilde{w}_x + \tilde{\epsilon}(\rho - \frac{1}{2}) - m^T \tilde{\xi}_3^T + n^T \tilde{\xi}_1^T &= 0, \\ m_t - \tilde{u}_x - (\rho - \frac{1}{2})\tilde{\xi}_3^T + (\tilde{\beta}^T + \tilde{\epsilon})m &= 0, \\ n_t - \tilde{v}_x - (\rho - \frac{1}{2})\tilde{\xi}_1^T + (\tilde{\gamma}^T + \tilde{\epsilon})n &= 0.\end{aligned}$$

对此方程组可求得如下解:

$$\begin{aligned}\tilde{\epsilon} &= w_x, \\ \tilde{\theta}_1 &= -w - 1, \quad \tilde{\theta}_2 = u, \quad \tilde{\theta}_3 = v, \\ \tilde{\xi}_3 &= -u_x^T, \quad \tilde{\xi}_1 = v_x^T, \\ \tilde{\beta} &= -\Pi(u, u_x), \quad \tilde{\gamma} = -\Pi(v, v_x), \\ \tilde{w} &= w\rho + u^T m + v^T n + \frac{1}{2}w - \frac{1}{2}, \\ \tilde{u} &= -\rho u - w m - \frac{1}{2}u, \\ \tilde{v} &= \rho v - (w + 2)n + \frac{1}{2}v,\end{aligned}$$

从而  $\rho(t, x)$ 、 $m(t, x)$  和  $n(t, x)$  满足以下  $(1+n+r)$  分量 Camassa-Holm 方程:

$$\begin{cases} \rho_t + 2w_x \rho + w\rho_x + \langle u, m \rangle_x + \langle u_x, m \rangle + \langle v, n \rangle_x + \langle v_x, n \rangle = 0, \\ m_t + 2w_x m + w m_x + 2\rho u_x + \rho_x u + \Pi(u, u_x)m = 0, \\ n_t + 2w_x n + (w + 2)n_x - 2\rho v_x - \rho_x v + \Pi(v, v_x)n = 0, \\ \rho = w - w_{xx}, \\ m = u - u_{xx}, \quad n = v - v_{xx}, \end{cases} \quad (7)$$

式中:  $w(t, x)$  是标量函数;  $u(t, x) = (u_1(t, x), \dots,$

$u_n(t, x))^T$  是  $n$  维函数;  $v(t, x) = (v_1(t, x), \dots, v_r(t, x))^T$  是  $r$  维函数;  $\Pi(v, v_x)$  和  $\Pi(u, u_x)$  表示两个  $n \times n$  反对称矩阵, 其中  $(\Pi(u, u_x))_{ij} = u_{i,x} u_j - u_i u_{j,x}$ ,  $i, j = 1, \dots, n$ . 方程(7)是  $(1+n+r)$  分量 KdV 方程(6)对偶可积系统.

## 2 $(1+n+r)$ 分量 CH 方程的尖峰孤波解

对  $t$ 、 $w$ 、 $u_i$  和  $v_i$  进行伸缩变换

$$t \mapsto \epsilon^{-1}t, \quad u_i \mapsto \epsilon u_i, \quad v_i \mapsto \epsilon v_i, \quad w \mapsto \epsilon w, \\ i = 1, \dots, n, \quad l = 1, \dots, r,$$

代入方程(7)得到以下方程:

$$\begin{cases} \rho_t + 2w_x \rho + w\rho_x + \langle u, m \rangle_x + \langle u_x, m \rangle + \langle v, n \rangle_x + \langle v_x, n \rangle = 0, \\ m_t + 2w_x m + w m_x + 2\rho u_x + \rho_x u + \epsilon \Pi(u, u_x)m = 0, \\ n_t + 2w_x n + (w + 2)n_x - 2\rho v_x - \rho_x v + \epsilon \Pi(v, v_x)n = 0. \end{cases}$$

令  $\epsilon \rightarrow 0$ , 可以得到以下方程:

$$\begin{cases} \rho_t + 2w_x \rho + w\rho_x + \langle u, m \rangle_x + \langle u_x, m \rangle + \langle v, n \rangle_x + \langle v_x, n \rangle = 0, \\ m_t + 2w_x m + w m_x + 2\rho u_x + \rho_x u = 0, \\ n_t + 2w_x n + (w + 2)n_x - 2\rho v_x - \rho_x v = 0, \\ \rho = w - w_{xx}, \\ m = u - u_{xx}, \quad n = v - v_{xx}. \end{cases} \quad (8)$$

本节将研究方程(8)柯西问题

$$\begin{cases} \rho_t + 2w_x \rho + w\rho_x + \langle u, m \rangle_x + \langle u_x, m \rangle + \langle v, n \rangle_x + \langle v_x, n \rangle = 0, \quad \rho = w - w_{xx}, \\ m_t + 2w_x m + w m_x + 2\rho u_x + \rho_x u = 0, \\ m = u - u_{xx}, \quad t > 0, x \in \mathbb{R}, \\ n_t + 2w_x n + (w + 2)n_x - 2\rho v_x - \rho_x v = 0, \\ n = v - v_{xx}, \quad t > 0, x \in \mathbb{R}, \\ w(0, x) = w_0(x), \\ u(0, x) = u_0(x), \quad u_0(x) = (u_{10}(x), \dots, u_{n0}(x))^T, \\ x \in \mathbb{R}, \\ v(0, x) = v_0(x), \quad v_0(x) = (v_{10}(x), \dots, v_{r0}(x))^T, \\ x \in \mathbb{R}, \\ w \rightarrow 0, \quad u \rightarrow 0, \quad v \rightarrow 0, \quad |x| \rightarrow \infty \end{cases} \quad (9)$$

尖峰孤立波解的存在性. 一般地, 将包含  $w$ 、 $u$  和  $v$  的表达式  $\rho$ 、 $m$  和  $n$  代入方程(9), 得到以下方程:

$$\begin{cases} w_t + ww_x + \langle \mathbf{u}, \mathbf{u}_x \rangle + \langle \mathbf{v}, \mathbf{v}_x \rangle + (1 - \partial_x^2)^{-1} \partial_x (w^2 + \\ \frac{1}{2} w_x^2 + \langle \mathbf{u}, \mathbf{u} \rangle + \frac{1}{2} \langle \mathbf{u}_x, \mathbf{u}_x \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \\ \frac{1}{2} \langle \mathbf{v}_x, \mathbf{v}_x \rangle) = 0, \\ \mathbf{u}_t + (w\mathbf{u})_x + (1 - \partial_x^2)^{-1} \partial_x (2w\mathbf{u} + w_x \mathbf{u}_x) = 0, \\ \mathbf{v}_t - (w\mathbf{v})_x + 2\mathbf{v}_x + (1 - \partial_x^2)^{-1} \partial_x (2w\mathbf{v} - w_x \mathbf{v}_x) - \\ 2(1 - \partial_x^2)^{-1} (w\mathbf{v}_x) = 0. \end{cases}$$

注意到

$$\begin{aligned} u &= (1 - \partial_x^2)^{-1} m = H * m, \\ v &= (1 - \partial_x^2)^{-1} n = H * n, \end{aligned} \quad (10)$$

式中:  $H(x) = e^{-|x|}/2$ ;  $*$  表示  $\mathbb{R}$  上的卷积. 现在定义方程(9)的弱解.

定义 给定初值  $w_0, u_{k0}, v_{h0} \in H_{\text{loc}}^1(\mathbb{R})$  ( $k = 1, \dots, n, h = 1, \dots, r$ ), 函数  $w, \mathbf{u}, \mathbf{v} \in L_{\text{loc}}^\infty([0, T], H_{\text{loc}}^1(\mathbb{R}))$  是初值问题(9)的弱解当且仅当它们满足以下方程:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} [w\phi_t + \frac{1}{2}(w^2 + \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle)\phi_x + H * (w^2 + \\ & \frac{1}{2} w_x^2 + \langle \mathbf{u}, \mathbf{u} \rangle + \frac{1}{2} \langle \mathbf{u}_x, \mathbf{u}_x \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \\ & \frac{1}{2} \langle \mathbf{v}_x, \mathbf{v}_x \rangle)\phi_x] dx dt + \int_{\mathbb{R}} w_0(x)\phi(0, x) dx = 0, \\ & \int_0^T \int_{\mathbb{R}} [u\phi_t + wu\phi_x + H * (2w\mathbf{u} + w_x \mathbf{u}_x)\phi_x] dx dt + \\ & \int_{\mathbb{R}} u_0(x)\phi(0, x) dx = 0, \\ & \int_0^T \int_{\mathbb{R}} [v\phi_t - wv\phi_x + 2v\phi_x + H * (2w\mathbf{v} - w_x \mathbf{v}_x)\phi_x - \\ & 2H * (w\mathbf{v}_x)] dx dt + \int_{\mathbb{R}} v_0(x)\phi(0, x) dx = 0, \end{aligned}$$

且任意测试函数  $\phi(t, x) \in C_c^\infty([0, T] \times \mathbb{R})$ . 如果对于任意  $T > 0$ ,  $w$ 、 $\mathbf{u}$  和  $\mathbf{v}$  是  $[0, T]$  上的弱解, 则它们被称为全局弱解.

定理 设任何  $a_0 \neq 0$ , 形式为

$$\begin{aligned} w(t, x) &= a_0 e^{-|x-ct|}, \\ \mathbf{u}(t, x) &= a e^{-|x-ct|}, \quad a = (a_1, \dots, a_n)^T, \\ \mathbf{v}(t, x) &= b e^{-|x-ct|}, \quad b = (b_1, \dots, b_r)^T, \end{aligned}$$

且满足条件

$$c = 2a_0, \langle a, a \rangle + \langle b, b \rangle = a_0^2 \quad (11)$$

的表达式是方程(9)的全局弱解.

证明 首先, 在分布  $S'(\mathbb{R})$  的意义下, 对任何  $t \in \mathbb{R}^+$  有

$$\begin{aligned} \partial_x w(t, x) &= -\text{sign}(x - ct)w(t, x), \\ \partial_x \mathbf{u}(t, x) &= -\text{sign}(x - ct)\mathbf{u}(t, x), \\ \partial_x \mathbf{v}(t, x) &= -\text{sign}(x - ct)\mathbf{v}(t, x). \end{aligned} \quad (12)$$

设  $w_0(x) := w(0, x)$ ,  $\mathbf{u}_0(x) := \mathbf{u}(0, x)$ ,  $\mathbf{v}_0(x) := \mathbf{v}(0, x)$ ,  $x \in \mathbb{R}$ . 则有

$$\begin{aligned} \lim_{t \rightarrow 0^+} \|w(t, \cdot) - w_0(\cdot)\|_{W^{1,\infty}} &= 0, \\ \lim_{t \rightarrow 0^+} \|u(k, \cdot) - u_{k0}(\cdot)\|_{W^{1,\infty}} &= 0, \quad k = 1, \dots, n, \\ \lim_{t \rightarrow 0^+} \|v(t, \cdot) - v_{h0}(\cdot)\|_{W^{1,\infty}} &= 0, \quad h = 1, \dots, r. \end{aligned} \quad (13)$$

相似地, 对于任何  $t > 0$ , 有

$$\begin{aligned} \partial_t w(t, x) &= c \text{sign}(x - ct)w(t, x) \in L^\infty(\mathbb{R}), \\ \partial_t \mathbf{u}(t, x) &= c \text{sign}(x - ct)\mathbf{u}(t, x) \in L^\infty(\mathbb{R}), \\ \partial_t \mathbf{v}(t, x) &= c \text{sign}(x - ct)\mathbf{v}(t, x) \in L^\infty(\mathbb{R}). \end{aligned} \quad (14)$$

因此, 根据式(12)~(14)可以推断出, 对于每个测试函数  $\phi(t, x) \in C_c^\infty([0, \infty) \times \mathbb{R})$ ,

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}} [w\phi_t + \frac{1}{2}(w^2 + \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle)\phi_x] dx dt + \\ & \int_{\mathbb{R}} w_0(x)\phi(0, x) dx = - \int_0^{+\infty} \int_{\mathbb{R}} \text{sign}(x - ct) \times \\ & (cw(t, x) - w^2(t, x) - \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle) \times \\ & \phi(t, x) dx dt, \end{aligned} \quad (15)$$

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}} (u\phi_t + wu\phi_x) dx dt + \int_{\mathbb{R}} u_0(x)\phi(0, x) dx = \\ & - \int_0^{+\infty} \int_{\mathbb{R}} \text{sign}(x - ct) (cu(t, x) - \\ & 2w(t, x)u(t, x)) \phi(t, x) dx dt, \end{aligned} \quad (16)$$

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}} (v\phi_t - wv\phi_x + 2v\phi_x) dx dt + \int_{\mathbb{R}} v_0(x) \times \\ & \phi(0, x) dx = - \int_0^{+\infty} \int_{\mathbb{R}} \text{sign}(x - ct) \times \\ & (cv(t, x) + 2w(t, x)v(t, x) - 2v(t, x)) \times \\ & \phi(t, x) dx dt. \end{aligned} \quad (17)$$

进一步利用  $w$ 、 $\mathbf{u}$  和  $\mathbf{v}$  的定义, 有

$$\begin{aligned} & \text{sign}(x - ct)(cw - w^2 - \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle) = \\ & \begin{cases} ca_0 e^{ct-x} - (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) e^{2(ct-x)}, & x > ct, \\ -ca_0 e^{x-ct} + (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) e^{2(x-ct)}, & x \leq ct, \end{cases} \end{aligned} \quad (18)$$

$$\begin{aligned} & \text{sign}(x - ct)(cu - 2wu) = \\ & \begin{cases} cae^{ct-x} - 2a_0 a e^{2(ct-x)}, & x > ct, \\ -cae^{x-ct} + 2a_0 a e^{2(x-ct)}, & x \leq ct \end{cases} \end{aligned} \quad (19)$$

和

$$\begin{aligned} \text{sign}(x-ct)(cv+2wv-2v) = \\ \begin{cases} (c-2)be^{ct-x} + 2a_0be^{2(ct-x)}, & x > ct, \\ -(c-2)be^{x-ct} - 2a_0be^{2(x-ct)}, & x \leq ct. \end{cases} \quad (20) \end{aligned}$$

另一方面, 由式(10)有

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}} H_x * [w^2 + \frac{1}{2}w_x^2 + \langle u, u \rangle + \frac{1}{2}\langle u_x, u_x \rangle + \langle v, v \rangle + \\ \frac{1}{2}\langle v_x, v_x \rangle] \phi_x dx dt = - \int_0^{+\infty} \int_{\mathbb{R}} H_x * [w^2 + \frac{1}{2}w_x^2 + \\ \langle u, u \rangle + \frac{1}{2}\langle u_x, u_x \rangle + \langle v, v \rangle + \frac{1}{2}\langle v_x, v_x \rangle] \phi dx dt. \end{aligned}$$

由  $H_x = -\text{sign}(x)e^{-|x|}/2$  可以得到

$$\begin{aligned} H_x * [w^2 + \frac{1}{2}w_x^2 + \langle u, u \rangle + \frac{1}{2}\langle u_x, u_x \rangle + \langle v, v \rangle + \\ \frac{1}{2}\langle v_x, v_x \rangle](t, x) = -\frac{3}{4} \int_{-\infty}^{+\infty} \text{sign}(x-y) \times \\ (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) e^{-|x-y|-2|y-ct|} dy. \quad (21) \end{aligned}$$

当  $x > ct$  时, 将式(21)右侧的积分区间拆分为三个部分:

$$\begin{aligned} H_x * [w^2 + \frac{1}{2}w_x^2 + \langle u, u \rangle + \frac{1}{2}\langle u_x, u_x \rangle + \langle v, v \rangle + \\ \frac{1}{2}\langle v_x, v_x \rangle](t, x) = -\frac{3}{4} \times \\ (\int_{-\infty}^{ct} + \int_{ct}^x + \int_x^{+\infty}) \text{sign}(x-y) \times \\ (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) e^{-|x-y|-2|y-ct|} dy =: \\ I_1^{(1)} + I_2^{(1)} + I_3^{(1)}. \end{aligned}$$

直接计算得

$$\begin{aligned} I_1^{(1)} &= -\frac{3}{4} \int_{-\infty}^{ct} (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) e^{-(x-y)-2(ct-y)} dy = \\ &= -\frac{1}{4} (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) e^{ct-x}, \\ I_2^{(1)} &= -\frac{3}{4} \int_{ct}^x (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) e^{-(x-y)-2(y-ct)} dy = \\ &= \frac{3}{4} (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) (e^{2(ct-x)} - e^{ct-x}), \\ I_3^{(1)} &= \frac{3}{4} \int_x^{+\infty} (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) e^{-(y-x)-2(y-ct)} dy = \\ &= \frac{1}{4} (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) e^{2(ct-x)}. \end{aligned}$$

则当  $x > ct$  时,

$$\begin{aligned} H_x * [w^2 + \frac{1}{2}w_x^2 + \langle u, u \rangle + \frac{1}{2}\langle u_x, u_x \rangle + \langle v, v \rangle + \\ \frac{1}{2}\langle v_x, v_x \rangle](t, x) = (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) \times \\ (e^{2(ct-x)} - e^{ct-x}). \quad (22) \end{aligned}$$

而对于  $x \leq ct$  时, 使用类似的计算方法, 有

$$\begin{aligned} H_x * [w^2 + \frac{1}{2}w_x^2 + \langle u, u \rangle + \frac{1}{2}\langle u_x, u_x \rangle + \langle v, v \rangle + \\ \frac{1}{2}\langle v_x, v_x \rangle](t, x) = (a_0^2 + \langle a, a \rangle + \langle b, b \rangle) \times \\ (e^{x-ct} - e^{2(x-ct)}). \quad (23) \end{aligned}$$

如果  $x > ct$ , 有

$$\begin{aligned} H_x * (2wu + w_x u_x)(t, x) = \\ 2a_0 a (e^{2(ct-x)} - e^{ct-x}), \quad (24) \end{aligned}$$

如果  $x \leq ct$ , 则有

$$\begin{aligned} H_x * (2wu + w_x u_x)(t, x) = \\ 2a_0 a (e^{x-ct} - e^{2(x-ct)}) \quad (25) \end{aligned}$$

和

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}} [H * (2wv - w_x v_x) \phi_x - 2H * (wv_x)] dx dt = \\ - \int_0^{+\infty} \int_{\mathbb{R}} [H_x * (2wv - w_x v_x) \phi + \\ 2H * (wv_x)] dx dt. \end{aligned}$$

利用类似的计算可推出, 如果  $x > ct$ , 有

$$\begin{aligned} H_x * (2wv - w_x v_x)(t, x) + 2H * (wv_x)(t, x) = \\ \frac{4}{3} a_0 b (e^{2(ct-x)} - e^{ct-x}), \quad (26) \end{aligned}$$

如果  $x \leq ct$ , 则有

$$\begin{aligned} H_x * (2wv - w_x v_x)(t, x) + 2H * (wv_x)(t, x) = \\ \frac{4}{3} a_0 b (e^{x-ct} - e^{2(x-ct)}). \quad (27) \end{aligned}$$

因此, 将式(18)代入式(15), 式(19)代入式(16), 式(20)代入式(17), 并使用式(22)~(27)可以得到式(11). 定理证明完毕.

### 3 结论

本文给出了一些多分量CH类方程的几何背景, 并研究了该方程的可积性和尖峰孤子解. 得到了多分量KdV方程(6), 利用Möbius几何得到一个 $(1+n+r)$ 分量Camassa-Holm方程(7)并求出了它的尖峰孤子解. 该几何由Möbius群定义, 该群是文献[14]中定义的几何双曲形式. 该方程其他形式的解以及相关内容还有待进一步研究.

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## A multi-component Camassa-Holm equation arising from Möbius geometry

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**Abstract:** In this paper, we study curve flows in general Möbius geometry, and prove that a class of multi-component Camassa-Holm equations are equivalent to an invariant curve flow in Möbius geometry, which is a multi-component generalization of the two-component Camassa-Holm equation, and can also be regarded as a dual integrable system of a class of multi-component KdV systems. We also obtain a peaked solution for a degradation case of multi-component Camassa-Holm equations.

**Key words:** invariant curve flow; Möbius geometry; Camassa-Holm equation; multi-component Camassa-Holm system; peaked solution

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