

DOI: 10.20098/j.cnki.1001-5132.2023.0701

耦合 MRT 方程的三哈密顿对偶系统

胡碧圆, 曹陈辰*

(宁波大学 数学与统计学院, 浙江 宁波 315211)

摘要: 本文将三哈密顿方法应用于耦合 Merola-Ragnisco-Tu 方程, 获得了新的离散可积系统, 并求出了它的线性谱问题、双哈密顿结构以及 Darboux-Bäcklund 变换. 进一步, 利用 Darboux-Bäcklund 变换求出了对偶系统的精确解.

关键词: 线性谱问题; 双哈密顿结构; Darboux-Bäcklund 变换; 精确解

中图分类号: O175.2

文献标志码: A

文章编号: 1001-5132 (2024) 01-0031-07

物理、化学和生物学中的许多现象都可以用非线性离散方程来建模. 近年来, 离散方程的研究受到了广泛关注^[1-3]. 目前, 研究者们已经获得了很多离散可积系统, 如 Toda 链^[4-5]、sine-Gordon 链^[6]、Ablowitz-Ladik 链、Merola-Ragnisco-Tu (MRT) 链^[7]等. 这些离散可积系统的哈密顿结构、Lax 对、精确解等都已得到充分研究. 因此, 寻找或者构造一些新的离散可积系统是一个重要的研究内容. 利用三哈密顿对偶方法^[8]、Bäcklund 变换^[9]、递归算子^[10]等方法可以构造出新的可积系统.

1996 年, Olver 等^[11]通过重组哈密顿算子得到 CH 方程是 KdV 方程的对偶方程的结论, 同时也得到了 mKdV 的对偶系统. 2013 年, Tian 等^[12]利用三哈密顿对偶方法发现 Wadati-Konno-Ichikawa 方程的对偶系统是 Song-Qu-Qiao 方程. 2017 年, Kang 等^[13]同样利用此方法构造了 Dispersive Water Wave 系统的对偶系统, 并求出了对偶系统的 Bäcklund 变换. 2021 年, Zhang 等^[14]首次将此方法用到离散可积系统上, 构造了 MRT 方程的对偶系统.

2010 年, Xu^[15]构造了耦合 MRT 方程, 并求出了其哈密顿结构, 通过达布变换给出了精确解. 2015 年, Xu^[16]进一步得到了与其相关的非等谱可积族. 本文将通过重组哈密顿算子构造一个新的离散四分量可积系统.

1 耦合 MRT 的对偶系统

耦合 MRT 方程可以写成以下形式:

$$\begin{cases} r_{n,t} = r_{n+1} - r_n^2 s_n, \\ s_{n,t} = r_n s_n^2 - s_{n-1}, \\ u_{n,t} = -r_{n+1} + r_n^2 s_n + u_{n+1} - 2r_n s_n u_n - r_n^2 w_n, \\ w_{n,t} = -r_n s_n^2 + s_{n-1} - w_{n-1} + 2r_n s_n w_n + s_n^2 u_n, \end{cases} \quad (1)$$

其中 r_n, s_n, u_n, w_n 是位势函数. 耦合 MRT 方程有 4×4 矩阵形式的谱问题:

$$E\phi = U_n\phi, \quad \phi_{n,t} = V_n\phi,$$

其中

$$U_n = \begin{pmatrix} \lambda + r_n s_n & r_n & r_n w_n + s_n u_n & u_n \\ s_n & 1 & w_n & 0 \\ 0 & 0 & \lambda + r_n s_n & r_n \\ 0 & 0 & s_n & 1 \end{pmatrix},$$
$$V_n = \begin{pmatrix} \frac{\lambda}{2} & r_n & -\frac{\lambda}{2} & -r_n + u_n \\ s_{n-1} & -\frac{\lambda}{2} & -s_{n-1} + w_{n-1} & \frac{\lambda}{2} \\ 0 & 0 & \frac{\lambda}{2} & r_n \\ 0 & 0 & s_{n-1} & -\frac{\lambda}{2} \end{pmatrix}.$$

收稿日期: 2023-07-04.

宁波大学学报(理工版)网址: <http://journal1g.nbu.edu.cn/>

基金项目: 国家自然科学基金(12101339).

第一作者: 胡碧圆, 硕士研究生, 主要研究方向: 可积系统. E-mail: 2273826965@qq.com

*通信作者: 曹陈辰, 博士, 主要研究方向: 群论. E-mail: caochenchen@nbu.edu.cn

算子 E 定义为

$$Ef_n = f_{n+1}, E^{-1}f_n = f_{n-1}, E^k f_n = f_{n+k}, \\ n, k \in \mathbb{Z}.$$

耦合 MRT 方程用双哈密顿结构表示为

$$\begin{pmatrix} r_n \\ s_n \\ u_n \\ w_n \end{pmatrix}_t = J_1 \begin{pmatrix} \frac{\delta H_2}{\delta r_n} \\ \frac{\delta H_2}{\delta s_n} \\ \frac{\delta H_2}{\delta u_n} \\ \frac{\delta H_2}{\delta w_n} \end{pmatrix} = J_2 \begin{pmatrix} \frac{\delta H_1}{\delta r_n} \\ \frac{\delta H_1}{\delta s_n} \\ \frac{\delta H_1}{\delta u_n} \\ \frac{\delta H_1}{\delta w_n} \end{pmatrix},$$

式中:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ J_2 = \begin{pmatrix} 0 & 0 & -r_n \theta r_n & \\ 0 & 0 & N_{23} & \\ -r_n \theta r_n & N_{32} & -r_n \theta u_n - u_n \theta r_n & \\ N_{41} & -s_n \theta s_n & N_{43} & \\ & N_{14} & & \\ & -s_n \theta s_n & & \\ & N_{34} & & \\ & -s_n \theta w_n - w_n \theta s_n & & \end{pmatrix} \rightarrow \\ \leftarrow \begin{pmatrix} N_{14} \\ -s_n \theta s_n \\ N_{34} \\ -s_n \theta w_n - w_n \theta s_n \end{pmatrix}, \\ H_1 = -\sum_{n \in \mathbb{Z}} (r_n s_n - r_n w_n - s_n u_n), \\ H_2 = -\frac{1}{2} \sum_{n \in \mathbb{Z}} (r_n s_{n-1} - r_n w_{n-1} - s_{n-1} u_n + r_{n+1} s_n - \\ s_n u_{n+1} - r_{n+1} w_n - r_n^2 s_n^2 + 2r_n s_n^2 u_n + 2r_n^2 s_n w_n),$$

其中

$$\theta = (1+E)(1-E)^{-1}, \\ N_{14} = N_{32} = E - r_n s_n + r_n \theta s_n, \\ N_{23} = N_{41} = -E^{-1} + r_n s_n + s_n \theta r_n, \\ N_{34} = u_n \theta s_n + r_n \theta w_n - r_n w_n - s_n u_n, \\ N_{43} = s_n \theta u_n + w_n \theta r_n + r_n w_n + s_n u_n.$$

首先, 分解 J_2 得到

$$J_2^{(1)} = \begin{pmatrix} 0 & 0 & 0 & E \\ 0 & 0 & -E^{-1} & 0 \\ 0 & E & 0 & 0 \\ -E^{-1} & 0 & 0 & 0 \end{pmatrix},$$

$$J_2^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -r_n \theta r_n & -r_n s_n + r_n \theta s_n \\ r_n s_n + s_n \theta r_n & -s_n \theta s_n \\ -r_n \theta r_n & -r_n s_n + r_n \theta s_n \\ r_n s_n + r_n \theta s_n & -s_n \theta s_n \\ -r_n \theta u_n - u_n \theta r_n & N_{34} \\ N_{43} & -s_n \theta w_n - w_n \theta s_n \end{pmatrix} \rightarrow \leftarrow$$

其中 $J_1, J_2^{(1)}, J_2^{(2)}$ 是相容的哈密顿算子. 显然, 对任意常数 a, b, c , 反对称算子 $J = aJ_1 + bJ_2^{(1)} + cJ_2^{(2)}$ 是哈密顿算子. 选择 $a = b = 1, c = 0$ 和 $a = b = 0, c = 1$, 可得到一对新的相容的哈密顿算子, 即 $\hat{J}_1 = J_1 + J_2^{(1)}, \hat{J}_2 = J_2^{(2)}$. 基于 $\hat{J}_1$ 和 $\hat{J}_2$ 可以得到耦合 MRT 方程的对偶系统. $\hat{J}_1$ 可以写成

$$\hat{J}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot S,$$

$$\text{其中 } S = \begin{pmatrix} 0 & 0 & 0 & 1+E \\ 0 & 0 & 1+E^{-1} & 0 \\ 0 & 1+E & 0 & 0 \\ 1+E^{-1} & 0 & 0 & 0 \end{pmatrix}. \text{ 接着, 利}$$

用 S 定义新的变量 $\hat{r}_n, \hat{s}_n, \hat{u}_n, \hat{w}_n$, 即

$$\begin{pmatrix} \hat{r}_n \\ \hat{s}_n \\ \hat{u}_n \\ \hat{w}_n \end{pmatrix} = S \begin{pmatrix} r_n \\ s_n \\ u_n \\ w_n \end{pmatrix} = \begin{pmatrix} w_n + w_{n+1} \\ u_n + u_{n-1} \\ s_n + s_{n+1} \\ r_n + r_{n-1} \end{pmatrix}. \quad (2)$$

所以 $\hat{J}_2$ 可以表示成

$$\hat{J}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\hat{r}_n \theta \hat{r}_n & -\hat{r}_n \hat{s}_n + \hat{r}_n \theta \hat{s}_n \\ \hat{r}_n \hat{s}_n + \hat{s}_n \theta \hat{r}_n & -\hat{s}_n \theta \hat{s}_n \\ & -\hat{r}_n \theta \hat{r}_n \\ & \hat{r}_n \hat{s}_n + \hat{s}_n \theta \hat{r}_n \\ & -\hat{r}_n \theta \hat{u}_n - \hat{u}_n \theta \hat{r}_n \\ & \hat{s}_n \theta \hat{u}_n + \hat{w}_n \theta \hat{r}_n + \hat{r}_n \hat{w}_n + \hat{s}_n \hat{u}_n \\ & -\hat{r}_n \hat{s}_n + \hat{r}_n \theta \hat{s}_n \\ & -\hat{s}_n \theta \hat{s}_n \\ & \hat{u}_n \theta \hat{s}_n + \hat{r}_n \theta \hat{w}_n - \hat{r}_n \hat{w}_n - \hat{s}_n \hat{u}_n \\ & -\hat{s}_n \theta \hat{w}_n - \hat{w}_n \theta \hat{s}_n \end{pmatrix} \rightarrow \leftarrow$$

一对相容的哈密顿算子 $\hat{J}_1, \hat{J}_2$ 可以生成 $\hat{r}_n$, $\hat{s}_n, \hat{u}_n, \hat{w}_n$ 的哈密顿系统, 所以有

$$\begin{pmatrix} \hat{r}_n \\ \hat{s}_n \\ \hat{u}_n \\ \hat{w}_n \end{pmatrix}_t = \hat{J}_1 \begin{pmatrix} \frac{\delta \hat{H}_m}{\delta \hat{r}_n} \\ \frac{\delta \hat{H}_m}{\delta \hat{s}_n} \\ \frac{\delta \hat{H}_m}{\delta \hat{u}_n} \\ \frac{\delta \hat{H}_m}{\delta \hat{w}_n} \end{pmatrix} = \hat{J}_2 \begin{pmatrix} \frac{\delta \hat{H}_{m-1}}{\delta \hat{r}_n} \\ \frac{\delta \hat{H}_{m-1}}{\delta \hat{s}_n} \\ \frac{\delta \hat{H}_{m-1}}{\delta \hat{u}_n} \\ \frac{\delta \hat{H}_{m-1}}{\delta \hat{w}_n} \end{pmatrix}, \quad m=1, 2, 3, \dots$$

接下来求解 $\hat{H}_1, \hat{H}_2$. 先利用种子方程和变分导数的链式法则, 可得到

$$\begin{pmatrix} \hat{r}_n \\ -\hat{s}_n \\ \hat{u}_n \\ -\hat{w}_n \end{pmatrix} = \hat{J}_1 \begin{pmatrix} \frac{\delta \hat{H}_1}{\delta \hat{r}_n} \\ \frac{\delta \hat{H}_1}{\delta \hat{s}_n} \\ \frac{\delta \hat{H}_1}{\delta \hat{u}_n} \\ \frac{\delta \hat{H}_1}{\delta \hat{w}_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\delta \hat{H}_1}{\delta \hat{r}_n} \\ \frac{\delta \hat{H}_1}{\delta \hat{s}_n} \\ \frac{\delta \hat{H}_1}{\delta \hat{u}_n} \\ \frac{\delta \hat{H}_1}{\delta \hat{w}_n} \end{pmatrix}.$$

因此, 通过计算可以得到

$$\begin{aligned} \hat{H}_1 &= r_n w_n + u_n s_n + \frac{1}{2} r_n w_{n+1} + \frac{1}{2} r_{n-1} w_n + \frac{1}{2} u_{n-1} s_n + \\ &\quad \frac{1}{2} u_n s_{n+1}, \end{aligned}$$

从而可以导出四分量 MRT 方程的对偶系统

$$\begin{pmatrix} \hat{r}_n \\ \hat{s}_n \\ \hat{u}_n \\ \hat{w}_n \end{pmatrix}_t = \begin{pmatrix} \hat{r}_n u_n (w_{n+1} - w_n) \\ \hat{s}_n w_n (u_n - u_{n-1}) \\ \hat{r}_n r_n (w_{n+1} - w_n) + \\ \hat{r}_n u_n (s_{n+1} - s_n) + \\ \hat{u}_n u_n (w_{n+1} - w_n) \\ \hat{s}_n w_n (r_n - r_{n-1}) + \\ \hat{s}_n s_n (u_n - u_{n-1}) + \\ \hat{w}_n w_n (u_n - u_{n-1}) \end{pmatrix}. \quad (3)$$

其次, 利用对偶系统的双哈密顿结构和方程 (3), 可以推出 $\hat{H}_2$:

$$\begin{aligned} \hat{H}_2 &= \frac{1}{2} \hat{r}_n r_n u_n (w_{n+1} - w_n) - \frac{1}{2} \hat{s}_n s_n w_n (u_n - u_{n-1}) + \\ &\quad \frac{1}{4} \hat{r}_n u_n^2 (s_{n+1} - s_n) + \frac{1}{4} \hat{u}_n u_n^2 (w_{n+1} - w_n) - \end{aligned}$$

$$\frac{1}{4} \hat{s}_n w_n^2 (r_n - r_{n-1}) - \frac{1}{4} \hat{w}_n w_n^2 (u_n - u_{n-1}).$$

可见, 本文得出了耦合 MRT 方程对偶系统的双哈密顿结构.

2 耦合 MRT 对偶系统的线性谱问题

首先, 根据耦合 MRT 方程的谱问题, 假设

$$\bar{U}_n = \begin{pmatrix} a_n & c_n & e_n & k_n \\ b_n & 1 & f_n & 0 \\ 0 & 0 & a_n & c_n \\ 0 & 0 & b_n & 1 \end{pmatrix},$$

$$\bar{V}_n = \begin{pmatrix} \alpha_n & \beta_n & -\sigma_n & \xi_n \\ \gamma_n & -\alpha_n & \eta_n & \sigma_n \\ 0 & 0 & \alpha_n & \beta_n \\ 0 & 0 & \gamma_n & -\alpha_n \end{pmatrix}.$$

利用相容性条件 $\bar{U}_{n,t} = (E \bar{V}_n) \bar{U}_n - \bar{U}_n \bar{V}_n$, 化简可得 $\alpha_n = (1-E)^{-1} (b_n \beta_n - c_n \gamma_{n+1})$, $\sigma_n = (1-E)^{-1} \times (k_n \gamma_{n+1} + c_n \eta_{n+1} - b_n \xi_n - f_n \beta_n)$, 从而有

$$\begin{pmatrix} c_n \\ b_n \\ k_n \\ f_n \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & -c_n \theta c_n \\ 0 & 0 & N'_{23} \\ -c_n \theta c_n & N'_{32} & -c_n \theta k_n - k_n \theta c_n \\ N'_{41} & -b_n \theta b_n & f_n \theta c_n + b_n \theta k_n + e_n \end{pmatrix} \rightarrow \begin{pmatrix} N'_{14} \\ -b_n \theta b_n \\ c_n \theta f_n + k_n \theta b_n - e_n \\ -f_n \theta b_n - b_n \theta f_n \end{pmatrix} \begin{pmatrix} \eta_{n+1} \\ \xi_n \\ \gamma_{n+1} \\ \beta_n \end{pmatrix}, \quad (4)$$

其中 $N'_{14} = N'_{32} = -a_n + c_n \theta b_n + E$, $N'_{23} = N'_{41} = a_n + b_n \theta c_n - E^{-1}$.

如果在矩阵 U_n 中令 $c_n = r_n$, $b_n = s_n$, $k_n = u_n$, $f_n = w_n$, $a_n = \lambda + r_n s_n$, $e_n = r_n w_n + s_n u_n$, 将其代入式 (4) 有

$$\begin{pmatrix} r_n \\ s_n \\ u_n \\ w_n \end{pmatrix}_t = (J_2 - \lambda J_1) \begin{pmatrix} \eta_{n+1} \\ \xi_n \\ \gamma_{n+1} \\ \beta_n \end{pmatrix}, \quad (5)$$

因此, 恢复了相容的哈密顿算子 J_1 和 J_2 . 为了得到耦合 MRT 方程, 需要选择

$$\eta_{n+1} = -s_n + w_n, \quad \xi_n = -r_n + u_n, \quad \gamma_{n+1} = s_n, \quad \beta_n = r_n.$$

将其代入 $\bar{V}_n$ 就导出了矩阵 V_n :

$$V_n = \begin{pmatrix} \frac{\lambda}{2} & r_n & -\frac{\lambda}{2} & -r_n + u_n \\ s_{n-1} & -\frac{\lambda}{2} & -s_{n-1} + w_{n-1} & \frac{\lambda}{2} \\ 0 & 0 & \frac{\lambda}{2} & r_n \\ 0 & 0 & s_{n-1} & -\frac{\lambda}{2} \end{pmatrix}.$$

其次, 为了获得耦合 MRT 方程对偶系统的谱问题, 先做变换 $r_n \rightarrow \lambda \hat{r}_n$, $s_n \rightarrow \lambda \hat{s}_n$, $u_n \rightarrow \lambda \hat{u}_n$, $w_n \rightarrow \lambda \hat{w}_n$, $\lambda \rightarrow -1$, 则式(5)变成

$$\lambda \begin{pmatrix} \hat{r}_n \\ \hat{s}_n \\ \hat{u}_n \\ \hat{w}_n \end{pmatrix}_t = (\hat{J}_1 + \lambda^2 \hat{J}_2) \begin{pmatrix} \eta_{n+1} \\ \xi_n \\ \gamma_{n+1} \\ \beta_n \end{pmatrix}. \quad (6)$$

这里就恢复了相容哈密顿算子 $\hat{J}_1$ 和 $\hat{J}_2$. 接着, 令

$$\eta_{n+1} = \frac{1}{\lambda} r_n, \xi_n = \frac{1}{\lambda} s_n, \gamma_{n+1} = \frac{1}{\lambda} u_n, \beta_n = \frac{1}{\lambda} w_n.$$

利用等式(2)计算式(6), 可以得到耦合 MRT 方程的对偶系统. 综上所述, 耦合 MRT 方程的对偶系统有满足零曲率方程的谱问题:

$$E\phi_n = \begin{pmatrix} \lambda^2 \hat{r}_n \hat{s}_n - 1 & \lambda \hat{r}_n & \lambda^2 \hat{r}_n \hat{w}_n + \lambda^2 \hat{s}_n \hat{u}_n & \lambda \hat{u}_n \\ \lambda \hat{s}_n & 1 & \lambda \hat{w}_n & 0 \\ 0 & 0 & \lambda^2 \hat{r}_n \hat{s}_n - 1 & \lambda \hat{r}_n \\ 0 & 0 & \lambda \hat{s}_n & 1 \end{pmatrix} \times \begin{pmatrix} \phi_n^1 \\ \phi_n^2 \\ \phi_n^3 \\ \phi_n^4 \end{pmatrix} = \hat{U}_n \phi_n, \quad (7)$$

$$\phi_{n,t} = \begin{pmatrix} u_{n-1} w_n - \frac{1}{2\lambda^2} & \frac{1}{\lambda} w_n \\ \frac{1}{\lambda} u_{n-1} & -u_{n-1} w_n + \frac{1}{2\lambda^2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} w_n r_{n-1} + s_n u_{n-1} & \frac{1}{\lambda} s_n \\ \frac{1}{\lambda} r_{n-1} & -w_n r_{n-1} - s_n u_{n-1} \\ u_{n-1} w_n - \frac{1}{2\lambda^2} & \frac{1}{\lambda} w_n \\ \frac{1}{\lambda} u_{n-1} & -u_{n-1} w_n + \frac{1}{2\lambda^2} \end{pmatrix} \times$$

$$\begin{pmatrix} \phi_n^1 \\ \phi_n^2 \\ \phi_n^3 \\ \phi_n^4 \end{pmatrix} = \hat{V}_n \phi_n. \quad (8)$$

3 Darbux-Bäcklund 变换

首先介绍 gauge 变换

$$\tilde{\phi}_n = T_n \phi_n,$$

这一变换将谱问题(7)和(8)变成

$$E\tilde{\phi}_n = \tilde{\phi}_{n+1} = \tilde{U}_n \tilde{\phi}_n, \quad \tilde{U}_n = T_{n+1} \hat{U}_n (T_n)^{-1}, \quad (9)$$

$$\tilde{\phi}_{n,t} = \tilde{V}_n \tilde{\phi}_n, \quad \tilde{V}_n = (T_{n,t} + T_n \hat{V}_n) (T_n)^{-1}. \quad (10)$$

令变换矩阵 T_n 为

$$T_n = \begin{pmatrix} t_{11}[n]\lambda + \frac{1}{\lambda} & t_{12}[n] \\ t_{21}[n] & t_{22}[n]\lambda + \frac{1}{\lambda} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} t_{11}[n]\lambda + \frac{1}{\lambda} & t_{12}[n] \\ t_{21}[n] & t_{22}[n]\lambda + \frac{1}{\lambda} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \leftarrow \begin{pmatrix} t_{13}[n]\lambda + \frac{1}{\lambda} & t_{14}[n] \\ t_{23}[n] & t_{24}[n]\lambda + \frac{1}{\lambda} \\ t_{11}[n]\lambda + \frac{1}{\lambda} & t_{12}[n] \\ t_{21}[n] & t_{22}[n]\lambda + \frac{1}{\lambda} \end{pmatrix},$$

其中 $t_{jl}[n]$ ($j=1,2, l=1,2,3,4$) 是关于变量 n 和 t 的未知函数.

令 $\phi_n = (\phi_n^1, \phi_n^2, \phi_n^3, \phi_n^4)^T$, $\psi_n = (\psi_n^1, \psi_n^2, \psi_n^3, \psi_n^4)^T$ 是谱问题(7)和(8)的线性独立的解, 假设 $\pm\lambda_1, \pm\lambda_2$ 是 $\det(T_n)=0$ 的根. 当 $\lambda = \pm\lambda_i$ ($i=1,2$) 时, 有 $T_n \phi_n$ 与 $T_n \psi_n$ 线性相关, 则存在非零常数 k_i ($i=1,2$), 使得

$$T_n \phi_n = k_i (T_n \psi_n), \quad \lambda = \lambda_i, \quad i=1,2. \quad (11)$$

首先, 定义

$$\begin{aligned} \alpha_i[n] &= \frac{\phi_n^2 - k_i \psi_n^2}{\phi_n^1 - k_i \psi_n^1}, \\ \beta_i[n] &= \frac{\phi_n^3 - k_i \psi_n^3}{\phi_n^1 - k_i \psi_n^1}, \\ \gamma_i[n] &= \frac{\phi_n^4 - k_i \psi_n^4}{\phi_n^1 - k_i \psi_n^1}, \\ \lambda &= \lambda_i, \quad i=1,2. \end{aligned} \quad (12)$$

因此, 基于式(12), 可以从式(11)中推知

$$\begin{aligned}
 t_{11}[n] &= \frac{\lambda_1 \beta_2 \gamma_1 - \lambda_2 \beta_1 \gamma_2}{\lambda_1 \lambda_2 (\lambda_1 \beta_1 \gamma_2 - \lambda_2 \beta_2 \gamma_1)}, \\
 t_{12}[n] &= \frac{(\lambda_1^2 - \lambda_2^2) \beta_1 \beta_2}{\lambda_1 \lambda_2 (\lambda_2 \beta_2 \gamma_1 - \lambda_1 \beta_1 \gamma_2)}, \\
 t_{13}[n] &= [(\lambda_1^2 - \lambda_2^2)(\gamma_1 \gamma_2 (\beta_1 - \beta_2) - \\
 &\quad \beta_1 \beta_2 (\alpha_2 \gamma_1 - \alpha_1 \gamma_2)) + \\
 &\quad (\lambda_1 \beta_2 \gamma_1 - \lambda_2 \beta_1 \gamma_2)(\lambda_1 \beta_1 \gamma_2 - \lambda_2 \beta_2 \gamma_1)] / \\
 &\quad [\lambda_1 \lambda_2 (\lambda_1 \beta_1 \gamma_2 - \lambda_2 \beta_2 \gamma_1)^2], \\
 t_{14}[n] &= [(\lambda_1^2 - \lambda_2^2) \beta_1 \beta_2 (\lambda_2 \beta_2 (\gamma_1 + \alpha_1) - \\
 &\quad \lambda_1 \beta_1 (\gamma_2 + \alpha_2)) - (\lambda_1^2 - \lambda_2^2)(\lambda_1 \beta_1^2 \gamma_2 - \\
 &\quad \lambda_2 \beta_2^2 \gamma_1)] / [\lambda_1 \lambda_2 (\lambda_2 \beta_2 \gamma_1 - \lambda_1 \beta_1 \gamma_2)^2], \quad (13) \\
 t_{21}[n] &= \frac{(\lambda_1^2 - \lambda_2^2) \gamma_1 \gamma_2}{\lambda_1 \lambda_2 (\lambda_2 \gamma_2 \beta_1 - \lambda_1 \gamma_1 \beta_2)}, \\
 t_{22}[n] &= \frac{\lambda_1 \gamma_2 \beta_1 - \lambda_2 \gamma_1 \beta_2}{\lambda_1 \lambda_2 (\lambda_1 \gamma_1 \beta_2 - \lambda_2 \gamma_2 \beta_1)}, \\
 t_{23}[n] &= [(\lambda_1^2 - \lambda_2^2) \gamma_1 \gamma_2 (\lambda_2 \gamma_2 (\beta_1 + 1) - \\
 &\quad \lambda_1 \gamma_1 (\beta_2 + 1)) - (\lambda_1^2 - \lambda_2^2)(\lambda_1 \alpha_2 \gamma_1^2 \beta_2 - \\
 &\quad \lambda_2 \alpha_1 \gamma_2^2 \beta_1)] / [\lambda_1 \lambda_2 (\lambda_2 \gamma_2 \beta_1 - \lambda_1 \gamma_1 \beta_2)^2], \\
 t_{24}[n] &= [(\lambda_1^2 - \lambda_2^2)(\beta_1 \beta_2 (\alpha_2 \gamma_1 - \alpha_1 \gamma_2) - \\
 &\quad \gamma_1 \gamma_2 (\beta_1 - \beta_2)) + (\lambda_1 \alpha_2 \beta_1 - \lambda_2 \alpha_1 \beta_2)(\lambda_1 \gamma_1 \beta_2 - \\
 &\quad \lambda_2 \gamma_2 \beta_1)] / [\lambda_1 \lambda_2 (\lambda_2 \gamma_2 \beta_1 - \lambda_1 \gamma_1 \beta_2)^2].
 \end{aligned}$$

因此,

$$\det(\mathbf{T}_n) = \frac{(\lambda^2 - \lambda_1^2)^2 (\lambda^2 - \lambda_2^2)^2}{\lambda^4 \lambda_1^4 \lambda_2^4}.$$

显然, $\pm \lambda_i$ ($i=1,2$) 是 $\det(\mathbf{T}_n)=0$ 的 4 个根. 接着, 利用空间谱问题(7)和等式(12)可以得到

$$\begin{aligned}
 \alpha_i[n+1] &= \frac{B_i[n]}{A_i[n]}, \\
 \beta_i[n+1] &= \frac{C_i[n]}{A_i[n]}, \\
 \gamma_i[n+1] &= \frac{D_i[n]}{A_i[n]},
 \end{aligned} \quad (14)$$

其中 $A_i[n] = \lambda_i^2 \hat{r}_n \hat{s}_n - 1 + \lambda_i \hat{r}_n \alpha_i + \lambda_i^2 \hat{r}_n \hat{w}_n \beta_i + \lambda_i^2 \hat{s}_n \hat{u}_n \beta_i + \lambda_i \hat{u}_n \gamma_i$, $B_i[n] = \lambda_i \hat{s}_n + \alpha_i + \lambda_i \hat{w}_n \beta_i$, $C_i[n] = \lambda_i^2 \hat{r}_n \hat{s}_n \beta_i - \beta_i + \lambda_i \hat{r}_n \gamma_i$, $D_i[n] = \lambda_i \hat{s}_n \beta_i + \gamma_i$.

将式(14)代入式(13), 很容易导出 $t_{jl}[n+1]$ ($j=1,2, l=1,2,3,4$) 与 $A_i[n], B_i[n], C_i[n], D_i[n]$ 之间的关系.

命题 1 由式(9)定义的矩阵 $\tilde{\mathbf{U}}_n$ 与 $\hat{\mathbf{U}}_n$ 有一样的

形式, 即

$$\tilde{\mathbf{U}}_n = \begin{pmatrix} \lambda^2 \tilde{r}_n \tilde{s}_n - 1 & \lambda \tilde{r}_n & \lambda^2 \tilde{r}_n \tilde{w}_n + \lambda^2 \tilde{s}_n \tilde{u}_n & \lambda \tilde{u}_n \\ \lambda \tilde{s}_n & 1 & \lambda \tilde{w}_n & 0 \\ 0 & 0 & \lambda^2 \tilde{r}_n \tilde{s}_n - 1 & \lambda \tilde{r}_n \\ 0 & 0 & \lambda \tilde{s}_n & 1 \end{pmatrix},$$

其中 $\tilde{r}_n, \tilde{s}_n, \tilde{u}_n, \tilde{w}_n$ 与 $\hat{r}_n, \hat{s}_n, \hat{u}_n, \hat{w}_n$ 的关系如下:

$$\begin{aligned}
 \tilde{r}_n &= \hat{r}_n + t_{12}[n] + t_{12}[n+1], \\
 \tilde{s}_n &= \hat{s}_n - t_{21}[n] - t_{21}[n+1], \\
 \tilde{u}_n &= \hat{u}_n + t_{14}[n] - t_{12}[n] + t_{14}[n+1] - t_{12}[n+1], \\
 \tilde{w}_n &= \hat{w}_n + t_{21}[n] - t_{23}[n] + t_{21}[n+1] - t_{23}[n+1].
 \end{aligned} \quad (15)$$

对于命题 1, 先由相容性得到

$$\mathbf{T}_{n+1} \hat{\mathbf{U}}_n \mathbf{T}_n^* = |\mathbf{T}_n| \mathbf{\Gamma}_n,$$

其中

$$\mathbf{\Gamma}_n = \begin{pmatrix} \lambda^2 \tau_{11} - 1 & \lambda \tau_{12} & \lambda^2 \tau_{13} & \lambda \tau_{14} \\ \lambda \tau_{21} & 1 & \lambda \tau_{23} & 0 \\ 0 & 0 & \lambda^2 \tau_{11} - 1 & \lambda \tau_{12} \\ 0 & 0 & \lambda \tau_{21} & 1 \end{pmatrix},$$

从而可以得到 $\mathbf{T}_{n+1} \hat{\mathbf{U}}_n = \mathbf{\Gamma}_n \mathbf{T}_n$. 再比较 λ 的同次幂系数, 其中由 λ 的零次幂系数可得公式(15), 之后借助于式(15)和 $t_{ij}[n], t_{ij}[n+1]$ 的表达式, 用 maple 验证其自动满足即可.

命题 2 在变换(15)下, 由式(10)定义的矩阵 $\tilde{\mathbf{V}}_n$ 与 $\hat{\mathbf{V}}_n$ 有相同的形式.

同理, 对于命题 2, 也是先由相容性得到

$$(\mathbf{T}_{n,t} + \mathbf{T}_n \hat{\mathbf{V}}_n) \mathbf{T}_n^* = |\mathbf{T}_n| \mathbf{A}_n,$$

其中

$$\mathbf{A}_n = \begin{pmatrix} \mu_{11} - \frac{1}{2\lambda^2} & \frac{1}{\lambda} \mu_{12} \\ \frac{1}{\lambda} \mu_{21} & -\mu_{22} + \frac{1}{2\lambda^2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \mu_{13} & \frac{1}{\lambda} \mu_{14} \\ \frac{1}{\lambda} \mu_{23} & -\mu_{24} \\ \mu_{11} - \frac{1}{2\lambda^2} & \frac{1}{\lambda} \mu_{12} \\ \frac{1}{\lambda} \mu_{21} & -\mu_{22} + \frac{1}{2\lambda^2} \end{pmatrix},$$

从而可以得到 $T_{n,t} + T_n \hat{V}_n = A_n T_n$. 之后, 比较 λ 的同次幂系数, 其中由 λ^{-2} 的系数可得

$$\begin{aligned}\mu_{12} &= w_n + t_{12}[n], \\ \mu_{21} &= u_{n-1} - t_{21}[n], \\ \mu_{14} &= r_{n-1} + t_{21}[n] - t_{23}[n], \\ \mu_{23} &= r_{n-1} + t_{21}[n] - t_{23}[n], \\ \mu_{11} &= \mu_{22} = \mu_{12}\mu_{21}, \\ \mu_{13} &= \mu_{24} = \mu_{12}\mu_{23} + \mu_{14}\mu_{21}.\end{aligned}$$

再利用等式(2)和变换(15), 可以得到 $\hat{V}_n$. 接下来用 maple 验证其余方程自动满足即可.

根据命题 1 和命题 2, 可以得到如下定理.

定理 1 等式(15)是方程(3)的 Darboux-Bäcklund 变换. 所以在变换(15)下, 方程(3)的任意一个解 $(\hat{r}_n, \hat{s}_n, \hat{u}_n, \hat{w}_n)^T$ 对应它的一个新解 $(\tilde{r}_n, \tilde{s}_n, \tilde{u}_n, \tilde{w}_n)^T$.

4 精确解

很容易验证 $r_n = 1, s_n = 1, u_n = 1, w_n = 1, \hat{r}_n = 2, \hat{s}_n = 2, \hat{u}_n = 2, \hat{w}_n = 2$ 满足方程, 这将作为种子解. 将这一种子解代入式(7)和式(8), 可以得到满足该方程组的两组解 $\phi_n = (\phi_n^1, \phi_n^2, \phi_n^3, \phi_n^4)^T$, $\psi_n = (\psi_n^1, \psi_n^2, \psi_n^3, \psi_n^4)^T$, 其中

$$\begin{aligned}\phi_n^1 &= 16e^{\frac{\sqrt{1+4\lambda^4}}{2\lambda^2}t} \lambda(1+4\lambda^4)^{-\frac{3}{2}} \times \\ &\left(\left(\left(\frac{1}{2}\lambda^4 + \frac{1}{16} \right) \sqrt{1+4\lambda^4} + \frac{\lambda^2}{4} + \lambda^6 \right) \times \right. \\ &\left. (-1)^n \left(\left(\sqrt{1+4\lambda^4} - 2\lambda^2 \right)^{n+2} + \left(\sqrt{1+4\lambda^4} - 2\lambda^2 \right)^{n+1} \right) + \right. \\ &\left. \left(\left(\frac{n-1}{2}\lambda^4 + \frac{2n-1}{16} \right) \sqrt{1+4\lambda^4} + (n-1)\lambda^6 + \right. \right. \\ &\left. \left. \frac{n-1}{4}\lambda^2 \right) \left(2\lambda^2 + \sqrt{1+4\lambda^4} \right)^{n-1} - \right. \\ &\left. \frac{4\lambda^4+1}{8} t \left(2\lambda^2 + \sqrt{1+4\lambda^4} \right)^{n+1} - \right. \\ &\left. \left(\left(\frac{1}{8}n + \frac{1}{2}\lambda^4 n + \frac{1}{16} \right) \sqrt{1+4\lambda^4} + \right. \right. \\ &\left. \left. \lambda^6 n - \frac{t}{2}\lambda^4 + \frac{n}{4}\lambda^2 - \frac{t}{8} \right) \left(2\lambda^2 + \sqrt{1+4\lambda^4} \right)^n \right),\end{aligned}$$

$$\begin{aligned}\phi_n^2 &= \left(\left(2\lambda^2 + \sqrt{1+4\lambda^4} \right)^n + \right. \\ &\left. (4\lambda^4+1)^{-1} \left(\left(8\lambda^4(n-1)\sqrt{1+4\lambda^4} + \right. \right. \right. \\ &\left. \left. 4\lambda^2(4\lambda^4+1)(n-1) + 2\lambda^2 \right) \times \right. \\ &\left. \left(2\lambda^2 + \sqrt{1+4\lambda^4} \right)^{n-1} + \right. \\ &\left. (-1)^n \left(2\lambda^2 + \sqrt{1+4\lambda^4} \right)^{-n-1} \times \right. \\ &\left. 8\lambda^4 \left(\sqrt{1+4\lambda^4} + \frac{8\lambda^4+1}{4\lambda^2} \right) \right) e^{\frac{\sqrt{1+4\lambda^4}}{2\lambda^2}t} + \\ &\frac{4\lambda^2 t \left(2\lambda^2 + \sqrt{1+4\lambda^4} \right)^n e^{\frac{\sqrt{1+4\lambda^4}}{2\lambda^2}t}}{\sqrt{1+4\lambda^4}}, \\ \phi_n^3 &= \frac{1}{2\lambda} \left(2\lambda^2 + \sqrt{4\lambda^4+1} - 1 \right) \times \\ &\left(2\lambda^2 + \sqrt{4\lambda^4+1} \right)^n e^{\frac{\sqrt{1+4\lambda^4}}{2\lambda^2}t}, \\ \phi_n^4 &= \left(2\lambda^2 + \sqrt{4\lambda^4+1} \right)^n e^{\frac{\sqrt{1+4\lambda^4}}{2\lambda^2}t}.\end{aligned}$$

$\psi_n = (\psi_n^1, \psi_n^2, \psi_n^3, \psi_n^4)^T$ 的形式与 ϕ_n 类似. 因此, 可以计算出 $\alpha_i[n], \beta_i[n], \gamma_i[n]$. 接着利用式(19)和式(20)可以求出 $t_{ji}[n+1]$ ($j=1, 2, l=1, 2, 3, 4$), 最后代入 Darboux-Bäcklund 变换(15), 就可以构造方程(3)的一组精确解.

参考文献:

- [1] TELA G Y, ZHANG D J. On a Special coupled lattice system of the discrete Boussinesq type[J]. Reports on Mathematical Physics, 2023, 91(2):219-235.
- [2] TELA G Y, ZHANG D J. Integrability and solutions for a fourth-order lattice Gel'fand-Dikii equation[J]. Applied Mathematics Letters, 2023, 135:108424.
- [3] QIN M L, WEN X Y. Integrable aspects, analytic solutions and their asymptotic analysis for a discrete relativistic Toda lattice system[J]. Advances in Continuous and Discrete Models, 2023, 2023(1):27.
- [4] TAKASAKI K. Initial value problem for the Toda lattice hierarchy[J]. Advanced Studies in Pure Mathematics, 1984, 4:139-163.
- [5] HIROTA R. Nonlinear partial difference equations. IV. Bäcklund transformation for the discrete-time Toda

- equation[J]. Journal of the Physical Society of Japan, 1978, 45(1):321-332.
- [6] HIROTA R. Nonlinear partial difference equations III; Discrete sine-Gordon equation[J]. Journal of the Physical Society of Japan, 1977, 43(6):2079-2086.
- [7] MEROLA I, RAGNISCO O, TU G Z. A novel hierarchy of integrable lattices[J]. Inverse Problems, 1994, 10(6): 1315-1334.
- [8] FUCHSSTEINER B. The Lie algebra structure of nonlinear evolution equations admitting infinite dimensional Abelian symmetry groups[J]. Progress of Theoretical Physics, 1981, 65(3):861-876.
- [9] NAKAMURA A, HIROTA R. Second modified KdV equation and its exact multi-soliton solution[J]. Journal of the Physical Society of Japan, 1980, 48(5):1755-1762.
- [10] OLVER P J. Evolution equations possessing infinitely many symmetries[J]. Journal of Mathematical Physics, 1977, 18(6):1212-1215.
- [11] OLVER P J, ROSENAU P. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support[J]. Physical Review E, 1996, 53(2): 1900.
- [12] TIAN K, LIU Q P. Tri-Hamiltonian duality between the Wadati-Konno-Ichikawa hierarchy and the Song-Qu-Qiao hierarchy[J]. Journal of Mathematical Physics, 2013, 54(4):043513.
- [13] KANG J, LIU X, OLVER P J, et al. Bäcklund transformations for tri-Hamiltonian dual structures of multi-component integrable systems[J]. Journal of Integrable Systems, 2017, 2(1):xyw016.
- [14] ZHANG M X, FENG B F, LIU J W. Tri-Hamiltonian duality system of Merola-Ragnisco-Tu equation[J]. Physics Letters A, 2021, 385:126966.
- [15] XU X X. An integrable coupling family of Merola-Ragnisco-Tu lattice systems, its Hamiltonian structure and related nonisospectral integrable lattice family[J]. Physics Letters A, 2010, 374(3):401-410.
- [16] XU X X. Solving an integrable coupling system of Merola-Ragnisco-Tu lattice equation by Darboux transformation of Lax pair[J]. Communications in Nonlinear Science and Numerical Simulation, 2015, 23(1/2/3):192-201.

Tri-Hamiltonian duality system of coupling Merola-Ragnisco-Tu equation

HU Biyuan, CAO Chenchen

(School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China)

Abstract: Applying the tri-Hamiltonian method to the coupled Merola-Ragnisco-Tu equation, a new discrete integral system is obtained, and its linear spectral problem, bi-Hamiltonian structure, and Darboux-Bäcklund transformation are solved. Furthermore, using the Darboux-Bäcklund transformation, the exact solution of the dual system can be sought.

Key words: linear spectral problem; bi-Hamiltonian structure; Darboux-Bäcklund transformation; exact solution

(责任编辑 韩 超)