

Research on Constraint and Double Wronskian for the Second-order AKNS Equation's Soliton Solutions

WAN Yun-zhen, LI Yu-qi*, LUO Si-long

(Faculty of Science, Ningbo University, Ningbo 315211, China)

Abstract: Based on the fact that the double Wronskian technique gives double Wronskian solutions of the second AKNS equation, and the restriction technique can be all soliton solution of the AKNS hierarchy of equations, the double Wronskian determinant sequence can be used to identify the link between these two types of solutions. The theorem proves that it meets constraint sequence by restriction technique, and the double Wronskian solution meets the corresponding sequence of the double Wronskian determinant.

Key words: AKNS equation; double Wronskian technique; soliton constraints

CLC number: O175

Document code: A

Article ID: 1001-5132 (2013) 04-0082-05

Various methods have been developed to search for multi-soliton solutions to Lax integrable systems. Each method has its physical background and benefits. Among them, many people have committed to the study of constraint methods^[1-3]. It had observed that reflection-less potential is some function of the eigenfunctions. With the observation, Cao^[1] suggested the method of nonlinearization of Lax pair. Then Li^[2] observed that the constraint between the potential and the eigenfunctions may also be regarded as symmetry constraint, and today the method of symmetry constraint method has become a powerful tool for analyzing the solutions of integrable systems.

Many physically important integrable partial differential equations belong to the AKNS hierarchy^[4-13]. The soliton solution of the AKNS hierarchy can be obtained by the inverse scattering method^[14]. The mathematical structures of the AKNS is still revealing^[15].

Recently, Li^[16] study the constraint of Lax integrable systems and he has discovered some interesting

things and his constraint may be or not be the symmetry constraint. They found the AKNS hierarchy can be constrained to a series of univariate integrable hierarchies and the key lied in finding a special form of Lax pair for the AKNS hierarchy. It proved that under the constraints all equations of AKNS hierarchy are linearizable. We can easily get the soliton solutions of the AKNS equation by it.

The other method is the Wronskian technique^[17-18]. Satsuma first cited that soliton solution can be expressed as Wronskian determinant form in 1979. However, he did not put this solution linked with the bilinear form of the soliton equations. Until 1983, Freeman^[17] proposed and built up this method. The determinant of Wronskian entries φ_i is the key to constructing Wronskian solutions.

Recently, Chen^[18] has got the Wronskian solutions of AKNS equation through the double Wronskian technique. It can be verified by the direct substitution into the solitons equation in the bilinear form.

Received date: 2013-01-04.

JOURNAL OF NINGBO UNIVERSITY (NSEE): <http://journal.nbu.edu.cn/>

Foundation items: Supported by the National Natural Science Foundation of China (11175092); Zhejiang Provincial Natural Science Foundation (R6090717); Ningbo Natural Science Foundation (2009B21003, 2010A610103, 2010A610095).

The first author: WAN Yun-zhen (1988-), female, Fuzhou Jiangxi, M.S. Candidate, research domain: mathematical physics. E-mail: yunzhen0721@163.com

***Corresponding author:** LI Yu-qi (1975-), male, Qingzhou Shandong, Doctor, research domain: mathematical physics. E-mail: liyuqi@nbu.edu.cn

As known to all, through a variety of different methods people can get different form of solutions. To some extent, these solutions may have a profound connection. In this paper, we will consider the relation between the form of soliton solution by constraint technique and the form of double Wronskian solution by double Wronskian technique of AKNS equation.

1 Double Wronskian solution of the AKNS equation

In this work, we will confine ourselves to the standard second-order isospectral AKNS equation^[19]:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \frac{1}{i} \begin{pmatrix} -q_{xx} + 2q^2 r \\ r_{xx} - 2qr^2 \end{pmatrix}. \quad (1)$$

The reason is that the soliton solutions of the standard AKNS have been greatly simplified by the work of Chen^[18] with the double Wronskian technique.

The Lax pair of Eq. (1) is

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = \begin{pmatrix} -2\eta^2 + qr & 2q\eta^2 - q_x \\ 2r\eta + r_x & 2\eta^2 - qr \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

where η is the eigenvalue parameter and n -order to obtain the double Wronskian solution for the AKNS (1), Chen^[18] first introduced two sets of auxiliary functions ϕ and ψ as:

$$\phi_x = -A\phi, \psi_x = A\psi, \quad (2)$$

$$\phi_t = 2i\phi_{xx}, \psi_t = -2i\psi_{xx}, \quad (3)$$

where A is an arbitrary constant $(n+m+2)(n+m+2)$ matrix, arbitrary real matrix independent of x and t ,

$$\phi = (\phi_1, \phi_2, \dots, \phi_{n+m+2})^T, \psi = (\psi_1, \psi_2, \dots, \psi_{n+m+2})^T.$$

They firstly proved that Eq. (1) has the following double Wronskian solution:

$$q = \frac{2W^{n+2,m}(\phi; \psi)}{W^{n+1,m+1}(\phi; \psi)}, r = \frac{-2W^{n,m+2}(\phi; \psi)}{W^{n+1,m+1}(\phi; \psi)}, \quad (4)$$

where $W^{j,l}(\phi; \psi)$ is defined as:

$$W^{j,l}(\phi; \psi) = |\phi, \partial\phi, \dots, \partial^{j-1}_x \phi; \psi, \partial\psi, \dots, \partial^{l-1}_x \psi|, \quad (5)$$

and j, l are any two positive integers. When j or l is equal to non-positive integer, then semicolon on the left part or right part does not exist, such as $l = 0, |j-1;$

$$|j-1| = |\hat{j}-1|.$$

2 The soliton solution of the AKNS by constraint

Definition 1^[15] Function p_k is completely determined by k :

$$p_{-1} = r, p_0 = 1, p_1 = q,$$

$$\frac{p_{k+1}}{p_k} = (qr - (\ln p_k)_{xx}) \frac{p_k}{p_{k-1}}. \quad (6)$$

Definition 2^[15] Function $\theta_k(g)$ is determined by both k and g :

$$\theta_1(g) = g, \theta_{k+1}(g) = (\ln(\theta_1(g)\theta_2(g)\cdots\theta_k(g)))_{xx} \theta_k(g). \quad (7)$$

Theorem 1^[15] The constraint $L^m q = 0$ is equivalent to $qr = (\ln p_m)_{xx}$, where $\hat{L} = (-\partial + q\partial^{-1}r)/i$, with the constraint the n -th AKNS equation is equivalent to $\partial u_m / \partial t = (-i)^{n-1} u_m^{(n)}$, where $u_m = p_{m-1} / p_m$ and p is defined by Eq. (6). Given u_m , q and r are determined by $q^{-1} = (-1)^{m-1} \theta_m(u_m)$ and $r = (-1)^m \theta_{m+1}(u_m)$ respectively, where θ is defined by Eq. (7).

According to the theorem 1, we can get the soliton solutions of second-order isospectral AKNS equation.

3 The relation between the two types of solution

Theorem 2 The double Wronskian solution Eq. (4) satisfies $qr = (\ln p_{m+1})_{xx}$, where

$$p_k = 2^{k^2} \sqrt{2} \cos\left(\frac{\pi}{4} + \frac{(k-1)\pi}{2}\right) \frac{|n+k; m-k|}{|\hat{n}; \hat{m}|}, \quad (8)$$

and p_k satisfies Eq. (6).

Theorem 2 means that each double Wronskian solution of the two-order AKNS is of invariant-manifold constraint type.

Before proving Theorem 2, we first introduce the well-known bilinear form of Eq. (1),

$$ff_{xx} - f_x^2 + gh = 0, \quad (9)$$

$$g_t f - gf_t - g_{xx} f - 2g_x f_x + gf_{xx} = 0, \quad (10)$$

$$h_t f - hf_t - h_{xx} f + 2h_x f_x - hf_{xx} = 0. \quad (11)$$

The solutions of (1) can be obtained by

$$q = \frac{g}{f}, r = \frac{h}{f}. \quad (12)$$

Chen^[20] proved that the bilinear form Eq. (13), Eq. (14) and Eq. (15) has the following double Wronskian solutions:

$$f = W^{n+1, m+1}(\phi; \psi) = |\hat{n}; \hat{m}|, \quad (13)$$

$$g = 2W^{n+2, m}(\phi; \psi) = 2|\hat{n}+1; \hat{m}-1|, \quad (14)$$

$$h = -2W^{n, m+2}(\phi; \psi) = -2|\hat{n}-1; \hat{m}+1|. \quad (15)$$

Accordingly Eq. (1) has solution Eq. (4). For convenience, we cite the following two lemmas.

Lemma 1^[14] If M is an $N \times (N-2)$ matrix and a, b, c, d are N -dimensional column vectors, then

$$|M, a, b||M, c, d| - |M, a, c||M, b, d| + \\ |M, a, d||M, b, c|. \quad (16)$$

Lemma 2^[18] Assume $P = (p_{i,j})$ is $l \times l$ operator matrix and its entries $p_{i,j}$ are differential operators. $B = (b_{i,j})$ is an $l \times l$ function matrix with column vector set b_i and row vector set b_j ($i=1, 2, \dots, l; j=1, 2, \dots, l$), then

$$\sum_{i=1}^l |b_1, \dots, p_i b_i, \dots, b_l| = \sum_{i=1}^l \begin{vmatrix} b_1' \\ \vdots \\ p_i b_i' \\ \vdots \\ b_l' \end{vmatrix}, \quad (17)$$

where

$$p_i b_i = (p_{1i} b_{1i}, p_{2i} b_{2i}, \dots, p_{li} b_{li})^T, \\ p_j' b_j' = (p_{j1}' b_{j1}', p_{j2}' b_{j2}', \dots, p_{jl}' b_{jl}')^T.$$

By Lemma 2, the following two identities can be proved^[8]:

$$\text{tr} A |\hat{j}; \hat{l}| = -|\hat{j}-1, \hat{j}+1; \hat{k}|, \quad (18)$$

$$(\text{tr} A)^2 |\hat{j}; \hat{l}| = |\hat{j}-2, \hat{j}, \hat{j}+1; \hat{k}| + |\hat{j}-1, \hat{j}+2; \hat{k}| - \\ 2|\hat{j}-1, \hat{j}+1; \hat{k}-1, \hat{k}+1| + |\hat{j}; \hat{k}-2, \hat{k}, \hat{k}+1| + \\ |\hat{j}; \hat{k}-1, \hat{k}+2|, \quad (19)$$

where A is a $(j+k+2) \times (j+k+2)$ constant matrix. If $k=-1$, then Equ. (18) and Equ. (19) become

$$\text{tr} A |\hat{j}| = -|\hat{j}-1, \hat{j}+1|, \quad (20)$$

$$(\text{tr} A)^2 |\hat{j}; \hat{l}| = |\hat{j}-2, \hat{j}, \hat{j}+1| + |\hat{j}-1, \hat{j}+2|. \quad (21)$$

In order to facilitate the writing process, we will

shorten $|\hat{n}+k; \hat{m}-k|/\text{dx}$ as $|\hat{n}+k; \hat{m}-k|_x$.

Proof of Theorem 2:

(1) $p_{-1} = r, p_0 = 1$, and $p_1 = q$.

Substituting $k=-1$ to Eq. (12), we obtain $p_{-1} = 2\sqrt{2} \cos(\pi/4 - \pi) |\hat{n}-1; \hat{m}+1|/|\hat{n}; \hat{m}| = r$.

Similarly, for $k=0, p_0=1$ and for $k=1, p_1=q$.

(2) $qr = (\ln p_{m+1})_{xx}$.

By Eq. (12), $qr = (\ln p_{m+1})_{xx}$ is equivalent to

$$\frac{gh}{f^2} = (\ln(|\hat{n}+\hat{m}+1|/f))_{xx}. \quad (22)$$

Eq. (22) is equivalent to

$$\frac{gh}{f^2} + \ln(f)_{xx} = (\ln|\hat{n}+\hat{m}+1|)_{xx}. \quad (23)$$

By Eq. (9), the left side of Eq. (23) is 0. The right side of Eq. (23) can be expanded as

$$|\hat{n}+\hat{m}+1|_x = |\hat{n}+\hat{m}, \hat{n}+\hat{m}+2|. \quad (24)$$

But,

$$|\hat{n}+\hat{m}+1|_x = |\hat{n}+\hat{m}, \hat{n}+\hat{m}+2|,$$

$$|\hat{n}+\hat{m}+1|_{xx} = |\hat{n}+\hat{m}-1, \hat{n}+\hat{m}+1, \hat{n}+\hat{m}+2| + \\ |\hat{n}+\hat{m}, \hat{n}+\hat{m}+3|.$$

By Eq. (20) and Eq. (21) with $j=\hat{n}+\hat{m}+1$, Eq. (24) is 0, i.e., the right side of Eq. (23) is also 0.

$$(3) \frac{p_{k+1}}{p_k} = (qr - (\ln p_k)_{xx}) \frac{p_k}{p_{k-1}}.$$

Substituting Eq. (8) to Eq. (6), using Eq. (9), Eq. (6) is equivalent to

$$|\hat{n}+\hat{k}; \hat{m}-\hat{k}|_{xx} |\hat{n}+\hat{k}; \hat{m}-\hat{k}| - 4|\hat{n}+\hat{k}+1; \hat{m}-\hat{k}-1| \cdot \\ |\hat{n}+\hat{k}-1; \hat{m}-\hat{k}+1| = |\hat{n}+\hat{k}; \hat{m}-\hat{k}|_x^2. \quad (25)$$

Utilizing the character of determinant we have

$$|\hat{n}+\hat{k}; \hat{m}-\hat{k}|_x = |\hat{n}+\hat{k}-1, \hat{n}+\hat{k}+1; \hat{m}-\hat{k}| + \\ |\hat{n}+\hat{k}; \hat{m}-\hat{k}-1, \hat{m}-\hat{k}+1|, \quad (26)$$

and

$$|\hat{n}+\hat{k}; \hat{m}-\hat{k}|_{xx} = |\hat{n}+\hat{k}-2, \hat{n}+\hat{k}, \hat{n}+\hat{k}+1; \hat{m}-\hat{k}| + \\ |\hat{n}+\hat{k}-1, \hat{n}+\hat{k}+2; \hat{m}-\hat{k}| + |\hat{n}+\hat{k}; \hat{m}-\hat{k}-2, \\ \hat{m}-\hat{k}, \hat{m}-\hat{k}+1| |\hat{n}+\hat{k}; \hat{m}-\hat{k}-1, \hat{m}-\hat{k}+2| + \\ 2|\hat{n}+\hat{k}-1, \hat{n}+\hat{k}+1; \hat{m}-\hat{k}-1, \hat{m}-\hat{k}+1|. \quad (27)$$

By Lemma 1, we have

$$\begin{aligned}
& |n+\hat{k}+1; m-\hat{k}-1| |n+\hat{k}-1; m-\hat{k}+1| = \\
& |n+\hat{k}; m-\hat{k}| |n+\hat{k}-1, n+k+1; m-\hat{k}-1, \\
& m-k+1| - |n+\hat{k}; m-\hat{k}-1, m-k+1| \cdot \\
& |n+\hat{k}-1, n+k+1; m-\hat{k}|. \quad (28)
\end{aligned}$$

Substituting Eq. (26), Eq. (27) and Eq. (28) into Eq. (25) and simplifying, we get Eq. (25) is equivalent to

$$\begin{aligned}
& (|n+\hat{k}-2, n+k, n+k+1; m-\hat{k}| + |n+\hat{k}-1, \\
& n+k+2; m-\hat{k}| + |n+\hat{k}; m-\hat{k}-2, m-k, \\
& m-k+1| + |n+\hat{k}; m-\hat{k}-1, m-k+2| - \\
& 2|n+\hat{k}-1, n+k+1; m-\hat{k}-1, m-k+1|) \cdot \\
& |n+\hat{k}; m-\hat{k}| = (|n+\hat{k}-1, n+k+1; m-\hat{k}| - \\
& |n+\hat{k}; m-\hat{k}-1, m-k+1|)^2. \quad (29)
\end{aligned}$$

By Eq. (18) and Eq. (19), we know that Eq. (29) is true. Therefore Eq. (25) is true. So Eq. (6) must be true.

Example Let $A = \text{diagonal}(0, 1, -1, 2, -2)$, Then by Eq. (2) and Eq. (3), we can set

$$\phi = \begin{pmatrix} 2 \\ e^{x+2it} \\ e^{-x+2it} \\ e^{2x+8it} \\ e^{-2x+8it} \end{pmatrix}, \psi = \begin{pmatrix} -8i \\ 4e^{-x-2it} \\ 3e^{x-2it} \\ 2e^{-2x-8it} \\ e^{2x-8it} \end{pmatrix}.$$

Suppose $n=1$ and $m=2$. Then by Eq. (12) we get

$$\begin{aligned}
p_{-1} &= r = \frac{p_{-1}^n}{p_{-1}^d}, p_0 = 1, p_1 = q = \frac{p_1^n}{p_1^d}, \\
p_2 &= \frac{p_2^n}{p_2^d}, p_3 = \frac{18432e^{6x+36it}}{p_3^d},
\end{aligned}$$

where

$$\begin{aligned}
p_{-1}^n &= -48e^{2x}(-8e^{2x+4it} - 6e^{6x+4it} + 3e^{8x+16it} + \\
& 6e^{16it} + 9ie^{4x}), \\
p_{-1}^d &= e^{4it}(36ie^{2x+12it} - 54e^{4x+16it} + 32e^{6x+4it} + \\
& 48e^{6x+28it} - 36e^{8x+16it} + 18ie^{10x+12it} - 3e^{12x+16it} - \\
& 8e^{16it} - 12ie^{4x} - 9ie^{8x}), \\
p_1^n &= -36e^{2x+12it} - 54ie^{4x+16it} + 32ie^{6x+4it} + \\
& 48ie^{6x+28it} - 36ie^{8x+16it} - 18e^{10x+12it} - \\
& 3ie^{12x+16it} - 8ie^{16it} + 12e^{4x} + 9e^{8x}, \\
p_2^d &= 36ie^{2x+12it} - 54e^{4x+16it} + 32e^{6x+4it} + \\
& 48e^{6x+28it} - 36e^{8x+16it} + 18ie^{10x+12it} -
\end{aligned}$$

$$\begin{aligned}
& 3e^{12x+16it} - 8e^{16it} - 12ie^{4x} - 9ie^{8x}, \\
p_2^n &= -24e^{2x+20it}(-16ie^{2x+12it} + 24e^{4x+16it} - \\
& 12ie^{6x+12it} + ie^{8x} + 2i), \\
p_2^d &= 36ie^{2x+12it} - 54e^{4x+16it} + 32e^{6x+4it} + \\
& 48e^{6x+28it} - 36e^{8x+16it} + 18ie^{10x+12it} - \\
& 3e^{12x+16it} - 8e^{16it} - 12ie^{4x} - 9ie^{8x}, \\
p_2^n &= -e^{4it}(-48e^{2x+16it} + 54ie^{4x+12it} + 96e^{4x+28it} - \\
& 192ie^{6x+24it} + 36ie^{8x+12it} + 72e^{8x+28it} - \\
& 24e^{10x+16it} + 3ie^{12x+12it} + 8ie^{12it} - 8ie^{6x}), \\
p_3^d &= -36e^{2x+12it} - 54ie^{4x+16it} + 32ie^{6x+4it} + \\
& 48ie^{6x+28it} - 36ie^{8x+16it} - 18e^{10x+12it} - \\
& 3ie^{12x+16it} - 8ie^{16it} + 12e^{4x} + 9e^{8x}.
\end{aligned}$$

It can be checked directly that $qr = (\ln p_3)_{xx}$ and p_{-1}, p_0, p_1, p_2 and that p_3 satisfy Eq. (6).

4 Summary and discussion

In this work, we find there are two types of solutions for AKNS equations. One is the double Wronskian solutions and the other is the solutions of the AKNS hierarchy by constraint. We discover the form of determinant of p_k fully meet the form of constraints sequence p_k , which has a good correspondence with the the AKNS equation's double Wronskian solutions and we have proved $qr = (\ln p_{m+1})_{xx}$. From it we know the double Wronskian solutions may be included in the solutions of the AKNS hierarchy by constraint. Whether the two types of solutions is essentially equivalent? The question should be studied further.

References:

- [1] Cao Cewen. Nonlinearization of the Lax system for AKNS hierarchy[J]. Science in China, Ser A, 1989, 33: 528-536.
- [2] Cheng Yi, Li Yishen. The constraint of the Kadomtsev-Petviashvili equation and its special solutions[J]. Phys Lett A, 1991, 157:22-26.
- [3] Zhou Zixiang. Finite dimensional Hamiltonian system related to a Lax pair with symplectic and cyclic symmetries[J]. Nonlinearity, 2012, 25:371-396.
- [4] Ablowitz M J, Kaup D J, Newell A C, et al. The inverse scattering transform: Fourier analysis for nonlinear

- problems[J]. Stud Appl Math, 1974, 53:249-255.
- [5] Lou Senyue. Nontruncated series symmetries and Generalized Virasoro algebra: AKNS-type breaking soliton equation[J]. J Phys A: Math Gen, 1995, 28:5943-5949.
- [6] Tang Xiaoyan, Lou Senyue. Variable separation solutions for the (2+1)-dimensional Burgers equations[J]. Chinese Phys Lett, 2003, 20(3):335-337.
- [7] Jia Ma, Wang Jianyong, Lou Senyue. Approximate symmetry reduction to the perturbed one-dimensional nonlinear Schrodinger equation[J]. Chinese Phys Lett, 2009, 26:1-4.
- [8] Zeng Yunbo. Solving AKNS equations via Jacobi inversion problem[J]. Acta Mathematicae Applicatae Sinica, 1999, 15(4):337-344.
- [9] Hao Honghai, Lu Lili, Chen Dengyuan. Double Wronskian soliton solution for mixed AKNS system[J]. Chinese Phys Lett, 2005, 22(12):2987-2990.
- [10] Yu Fajun. Noncommutative AKNS equation Hierarchy and its integrable couplings with Kronecker product[J]. Chinese Phys Lett, 2008, 25(2):359-362.
- [11] Yu Fajun. R-matrix of the coupling AKNS equation Hierarchy[J]. Chinese Phys Lett, 2008, 25(1):3519-3523.
- [12] Steudel H, Kaup D J. Inverse scattering for an AKNS problem with rational reflection coefficients[J]. Inverse Problems, 2008, 24:25015-25019.
- [13] He Jingsong, Zhang Ling, Cheng Yi, et al. Determinant representation of Darboux transformation AKNS system [J]. Science in China Series A Mathematics, 2006, 36(9): 971-983.
- [14] Chen Dengyuan. Introduction to soliton theory[M]. Beijing: Science Press, 2006.
- [15] Li Nianhua, Li Yuqi. Constraints and soliton solutions for KdV Hierarchy and AKNS Hierarchy[J]. Commun Theor Phys, 2011, 56:609-613.
- [16] Li Yuqi, Li Biao and Lou Senyue. Constraints for evolution equations with some special form of Lax Pairs and distinguishing Lax Pairs by available constraints[J]. [EB/OL]. [210-10-19]. <http://arxiv.org/pdf/1008.1375v2.pdf>.
- [17] Freeman N C, Nimmo J J. Soliton solutions of the KdV and KP equations: The Wronskian technique[J]. Phys Lett A, 1983, 95:1-3.
- [18] Chen Dengyuan, Zhang Dajun, and Bi Jinbo. New double Wronskian solutions of the AKNS equation[J]. Science in China Series A: Mathematics, 2008, 51(1):57-59.
- [19] Li Yishen. Soliton and integrable system[M]. Shanghai: Shanghai Scientific and Technological Education Publishing House, 1999.

基于约束和双朗斯基的二阶 AKNS 方程的孤子解研究

万云珍, 李玉奇*, 罗嗣龙

(宁波大学 理学院, 浙江 宁波 315211)

摘要: 鉴于双朗斯基技术给出了二阶 AKNS 方程的双朗斯基解, 而约束技术又可以得到 AKNS 方程族一切孤子解的情况, 因此, 通过双朗斯基行列式序列找出了这两类解之间的联系, 并通过定理证明其满足由约束技术得到的约束序列, 且双朗斯基解也满足相应的双朗斯基行列式序列.

关键词: AKNS 方程; 双朗斯基技术; 孤子约束方法

(责任编辑 章践立)