

m 重对角因子循环矩阵求逆和广义逆的 Euclid 算法

侯东杰, 岑建苗*

(宁波大学 理学院, 浙江 宁波 315211)

摘要: 利用多项式 Euclid 算法给出了非奇异 m 重对角因子循环矩阵求逆的一个新算法, 并将该算法推广至求 m 重对角因子循环矩阵的群逆和 Moore-Penrose 逆, 及给出了具体的求逆步骤.

关键词: m 重对角因子循环矩阵; 逆; 群逆; Moore-Penrose 逆; Euclid 算法

中图分类号: O151.21

文献标志码: A

文章编号: 1001-5132 (2014) 02-0101-05

循环矩阵及其推广在数学、物理学和工程技术等领域应用广泛, 因此探讨 m 重对角因子循环矩阵其性质和应用显得很有必要. 笔者利用多项式的 Euclid 算法给出了 m 重对角因子循环矩阵求逆和广义逆的一种快速算法, 它的一个显著特点是无需预先知道该矩阵是否非奇异.

设 $R_t = D_t C_t$ 是一个非奇异的基本对角因子循环矩阵, 其中,

$$D_t = \text{diag}(d_{t,1}, d_{t,2}, \dots, d_{t,n_t}),$$

$$C_t = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n_t \times n_t}, \quad t=1, \dots, m.$$

定义 1^[1] 若复数域上 1 个 $n_1 n_2 \dots n_m \times n_1 n_2 \dots n_m$ 矩阵 A 满足 $A(R_1 \otimes R_2 \otimes \dots \otimes R_m) = (R_1 \otimes R_2 \otimes \dots \otimes R_m)A$, 则称 A 为 m 重对角因子循环矩阵. 用 $SC_{R_1, \dots, R_m}^{n_1, \dots, n_m}$ 表示复数域上所有与 $\bigotimes_{t=1}^m R_t$ 可交换的 m 重对角因子循环矩阵.

由文献[1]可知, R_t 极小多项式为 $x^{n_t} - \prod_{j=1}^{n_t} d_{t,j}$

($t=1, \dots, m$). 方便起见, 记首行元素为 $(a_{0, \dots, 0}, \dots, a_{j_1, \dots, j_m}, \dots, a_{n_1-1, \dots, n_m-1})$ 的 m 重对角因子循环矩阵为 $A = SC_{R_1, \dots, R_m}^{n_1, \dots, n_m}(a_{0, \dots, 0}, \dots, a_{j_1, \dots, j_m}, \dots, a_{n_1-1, \dots, n_m-1})$, 则 A 可表示为:

$$A = f(R_1, \dots, R_m) = \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \dots \sum_{j_m=0}^{n_m-1} a_{j_1, \dots, j_m} \prod_{t=1}^m \left(\prod_{s=0}^{j_t} d_{t,s} \right)^{-1} \bigotimes_{t=1}^m R_t^{j_t},$$

其中,

$$f(x_1, \dots, x_m) = \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \dots \sum_{j_m=0}^{n_m-1} a_{j_1, \dots, j_m} \prod_{t=1}^m \left(\prod_{s=0}^{j_t} d_{t,s} \right)^{-1} \prod_{t=1}^m x_t^{j_t},$$

称多项式 $f(x_1, \dots, x_m)$ 为 m 重对角因子循环矩阵 A 的表示多项式.

令:

$$d_t = \sqrt[n_t]{\prod_{s=0}^{n_t} d_{t,s}}, d_{t,0} = 1, t=1, 2, \dots, m,$$

$$\Delta_t = \text{diag}(\sigma_{t,1}, \sigma_{t,2}, \dots, \sigma_{t,n_t}),$$

其中, σ_{t,j_t} 满足以下的递推公式: $\sigma_{t,j_t+1} = (d_t / d_{t,j_t}) \cdot \sigma_{t,j_t}, 1 \leq j_t \leq n_t, \sigma_{t,n_t+1} = \sigma_{t,1}, t=1, \dots, m$.

令 $\omega_t = \exp(2\pi i / n_t)$ 为 n_t 次本原单位根, $F_t = (F_{i,j_t})_{n_t}$ 是 Fourier 矩阵, 其中 $F_{i,j_t} = \omega^{(i-1)(j_t-1)} / \sqrt{n_t}, 1 \leq i, j_t \leq n_t, t=1, 2, \dots, m$.

引理 1^[1] 设 $A = SC_{R_1, \dots, R_m}^{n_1, \dots, n_m}(a_{0, \dots, 0}, \dots, a_{j_1, \dots, j_m}, \dots, a_{n_1-1, \dots, n_m-1}) \in SC_{R_1, \dots, R_m}^{n_1, \dots, n_m}$, 则

$$\left(\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t \right)^{-1} A \left(\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t \right) = \text{diag}(f(d_1, \dots, d_m), \dots, f(d_1 \omega_1^{n_1-1}, \dots, d_m \omega_m^{n_m-1})),$$

并且, $A \in SC_{R_1, \dots, R_m}^{n_1, \dots, n_m}$ 当且仅当

$$\left(\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t \right)^{-1} A \left(\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t \right),$$

收稿日期: 2013-03-11.

宁波大学学报(理工版)网址: <http://journallg.nbu.edu.cn/>

基金项目: 浙江省自然科学基金(Y604089).

第一作者: 侯东杰(1987-), 女, 河南焦作人, 在读硕士研究生, 主要研究方向: 矩阵论. E-mail: 872693306@qq.com

*通信作者: 岑建苗(1959-), 男, 浙江宁波人, 教授, 主要研究方向: 矩阵论. E-mail: cenjianmiao@nbu.edu.cn

是对角矩阵.

引理 2^[1] 设 $A = SCACIRC_{R_1 \cdots R_m}^{n_1 \cdots n_m}(a_{0, \cdots, 0}, \cdots, a_{j_1, \cdots, j_m}, \cdots, a_{n_1-1, \cdots, n_m-1}) \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$, 则 A 的特征值为:

$$\lambda_{i_1 i_2 \cdots i_m} = f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) = \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \cdots \sum_{j_m=0}^{n_m-1} \prod_{t=1}^m (\prod_{s=0}^{j_t} d_{t,s})^{-1} a_{j_1, j_2, \cdots, j_m} \prod_{t=1}^m (d_t \omega_t^{i_t})^{j_t},$$

$$i_t = 0, \cdots, n_t - 1, t = 1, \cdots, m.$$

其中, $d_t, d_t \omega_t, d_t \omega_t^2, \cdots, d_t \omega_t^{n_t-1}$ 是 $x^{n_t} - \prod_{j_t=1}^{n_t} d_{t,j_t}$ ($t = 1, \cdots, m$) 的 n_t 个不同的根.

引理 3^[1] 设 $A, B \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$, 则 $AB = BA \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$, 若 A 非奇异, 则 $A^{-1} \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$.

设 $A = SCACIRC_{R_1 \cdots R_m}^{n_1 \cdots n_m}(a_{0, \cdots, 0}, \cdots, a_{j_1, \cdots, j_m}, \cdots, a_{n_1-1, \cdots, n_m-1})$ 是复数域上的 m 重对角因子循环矩阵, 则 A 谱为:

$$\sigma(A) = \{\lambda_{i_1 i_2 \cdots i_m} \mid i_t = 0, \cdots, n_t - 1, t = 1, \cdots, m\}.$$

设当 $\lambda_{i_1 i_2 \cdots i_m} = 0$ 时, $T_{i_1 \cdots i_m} = 0$; 当 $\lambda_{i_1 i_2 \cdots i_m} \neq 0$ 时, $T_{i_1 \cdots i_m} = 1 / \lambda_{i_1 i_2 \cdots i_m}$, $i_t = 0, \cdots, n_t - 1, t = 1, \cdots, m$.

如果令

$$B = (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) \text{diag}(T_{0 \cdots 0}, \cdots,$$

$$T_{i_1 \cdots i_m}, \cdots, T_{(n_1-1) \cdots (n_m-1)}) (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t)^{-1},$$

则由文献[1]可知 B 是 A 的谱逆, 即 $B = A^s$.

由于 $SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$ 中的任意矩阵 A 都有指标 1, 故 A 的谱逆 A^s 也是 A 的群逆. 进一步, 如果 R_t 是正规的, 则 $A^s = A^+$. 我们需要如下的定理.

定理 1^[1] 设 $A = SCACIRC_{R_1 \cdots R_m}^{n_1 \cdots n_m}(a_{0, \cdots, 0}, \cdots, a_{j_1, \cdots, j_m}, \cdots, a_{n_1-1, \cdots, n_m-1}) \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$ 是一个奇异的 m 重对角因子循环矩阵, 则 $A^\# = A^s \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$, 不失一般性, 假设:

$$A = (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) \text{diag}(f(d_1, \cdots, d_m), \cdots, f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}), \cdots, f(d_1 \omega_1^{k_1-1}, \cdots, d_m \omega_m^{k_m-1}), 0, \cdots, 0) (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t)^{-1}$$

其中, $f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) \neq 0, i_t = 0, \cdots, k_t - 1, t = 1, \cdots, m$, 则,

$$A^\# = A^s = (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) \text{diag}(\frac{1}{f(d_1, \cdots, d_m)}, \cdots, \frac{1}{f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})}, \cdots, \frac{1}{f(d_1 \omega_1^{k_1-1}, \cdots, d_m \omega_m^{k_m-1})}, \cdots, \frac{1}{f(d_1 \omega_1^{k_1-1}, \cdots, d_m \omega_m^{k_m-1})}),$$

$$0, \cdots, 0) (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t)^{-1}.$$

因此, 如果 R_t 是正规的, 则 $A^s = A^+ \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$.

1 主要结果

定理 2 设 $A = SCACIRC_{R_1 \cdots R_m}^{n_1 \cdots n_m}(a_{0, \cdots, 0}, \cdots, a_{j_1, \cdots, j_m}, \cdots, a_{n_1-1, \cdots, n_m-1}) \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$, $g_{R_t}(x)$ 是 R_t 的极小多项式, $t = 1, \cdots, m$, 则 $g_{R_1}(x), \cdots, g_{R_m}(x)$ 的根组 $(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})$, $i_t = 0, \cdots, n_t - 1, t = 1, \cdots, m$ 中有 k 个组是 $f(x_1, \cdots, x_m)$ 的零点的充要条件是 $\text{rank} A = n_1 n_2 \cdots n_m - k$, 其中, $f(x_1, \cdots, x_m)$ 是 A 的表示多项式.

证明 由引理 1 可知,

$$(\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t)^{-1} A (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) = D,$$

其中, $D = \text{diag}(f(d_1, d_2, \cdots, d_m), \cdots, f(d_1 \omega_1^{n_1-1}, \cdots, d_m \omega_m^{n_m-1}))$.

$$d_t, d_t \omega_t, d_t \omega_t^2, \cdots, d_t \omega_t^{n_t-1} \text{ 是 } g_{R_t}(x) = x^{n_t} - \prod_{j_t=1}^{n_t} d_{t,j_t},$$

($t = 1, \cdots, m$) 的 n_t 个不同的根, 所以,

$$A (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) = (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) D.$$

由于 $\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t$ 是非奇异的, 因此,

$$\text{rank} A = \text{rank} A (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) =$$

$$\text{rank} (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) D = \text{rank} D = D,$$

中不为 0 的个数.

如果 $g_{R_1}(x), \cdots, g_{R_m}(x)$ 的根组 $(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})$ 中有 k 个组是 $f(x_1, \cdots, x_m)$ 的零点, 即有 k 个 $f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) = 0$, 则有 $n_1 n_2 \cdots n_m - k$ 个 $f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) \neq 0$, 因此有 $\text{rank} A = n_1 n_2 \cdots n_m - k$.

反之, 如果 $\text{rank} A = n_1 n_2 \cdots n_m - k$, 则 D 中有 $n_1 n_2 \cdots n_m - k$ 个 $f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) \neq 0$. 因此 D 中有 k 个 $f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) = 0$, 即 $g_{R_1}(x), \cdots, g_{R_m}(x)$ 的根组 $(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})$ 中有 k 个组是 $f(x_1, \cdots, x_m)$ 的零点.

推论 1 设 $A = SCACIRC_{R_1 \cdots R_m}^{n_1 \cdots n_m}(a_{0, \cdots, 0}, \cdots, a_{j_1, \cdots, j_m}, \cdots, a_{n_1-1, \cdots, n_m-1}) \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$, 则 A 非奇异的充要条件是 $(f(x_1, \cdots, x_m), x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1,j_1}, \cdots, x_m^{n_m} - \prod_{j_m=1}^{n_m} d_{m,j_m}) = 1$, 其

中, $f(x_1, \cdots, x_m)$ 是 A 的表示多项式, $x_t^{n_t} - \prod_{j_t=1}^{n_t} d_{t,j_t}$ 是 R_t 的极小多项式 ($t = 1, \cdots, m$).

定理 3 设 $A = SCACIRC_{R_1 \cdots R_m}^{n_1 \cdots n_m}(a_{0, \cdots, 0}, \cdots, a_{j_1, \cdots, j_m}, \cdots, a_{n_1-1, \cdots, n_m-1}) \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$ 是 1 个非奇异的 m 重对角因子循环矩阵, A 的表示多项式为 $f(x_1, \cdots, x_m)$, 则存在次数最低的多项式 $u(x_1, \cdots, x_m)$ 满足:

$$u(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) = \frac{1}{f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})},$$

$$i_t = 0, 1, \cdots, n_t - 1, \quad t = 1, 2, \cdots, m,$$

其中, R_t 的特征多项式为 $x^{n_t} - \prod_{j_t=1}^{n_t} d_{t,j_t}$, 且 $A^{-1} = u(R_1, R_2, \cdots, R_m) \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$.

证明 由于 A 是一个非奇异的 m 重对角因子循环矩阵, 由推论 1 可知,

$$(f(x_1, \cdots, x_m), x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1,j_1}, \cdots, x_m^{n_m} - \prod_{j_m=1}^{n_m} d_{m,j_m}) = 1,$$

于是存在 $u', v_1, v_2, \cdots, v_m \in C[x_1, x_2, \cdots, x_m]$ 满足:

$$u'f(x_1, \cdots, x_m) + v_1(x_1^{n_1} - \prod_{j_1=1}^{n_1} d_{1,j_1}) + v_2(x_2^{n_2} - \prod_{j_2=1}^{n_2} d_{2,j_2}) + \cdots + v_m(x_m^{n_m} - \prod_{j_m=1}^{n_m} d_{m,j_m}) = 1,$$

其中, $C[x_1, x_2, \cdots, x_m]$ 为复数域上多元多项式环.

用 R_t 分别代替上式中的 x_t , 因为,

$$R_t^{n_t} - \prod_{j_t=1}^{n_t} d_{t,j_t} I_{n_t} = 0, \quad t = 1, \cdots, m,$$

于是,

$$u'(R_1, R_2, \cdots, R_m) f(R_1, R_2, \cdots, R_m) = I_{n_t}.$$

设 $u(x_1, x_2, \cdots, x_m)$ 为次数最低的多元多项式且满足:

$$u(x_1, x_2, \cdots, x_m) - u'(x_1, x_2, \cdots, x_m) \in \langle G_{R_1 R_2 \cdots R_m}(x_1, x_2, \cdots, x_m) \rangle,$$

其中, $G_{R_1 R_2 \cdots R_m}(x_1, x_2, \cdots, x_m)$ 定义为满足当 $x_1 = d_1, d_1 \omega_1, \cdots, d_1 \omega_1^{n_1-1}; x_2 = d_2, d_2 \omega_2, \cdots, d_2 \omega_2^{n_2-1}; \cdots; x_m = d_m, d_m \omega_m, \cdots, d_m \omega_m^{n_m-1}$ 时 $G_{R_1 R_2 \cdots R_m}(x_1, x_2, \cdots, x_m) = 0$ 的次数最低的首一多元多项式, 即对 $g_{R_1}(x), \cdots, g_{R_m}(x)$ 的 $n_1 n_2 \cdots n_m$ 个根组 $(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})$ $i_t = 1, \cdots, n_t, t = 1, \cdots, m$ 均有 $G_{R_1 R_2 \cdots R_m}(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) = 0$, 其中, 我们称 $\langle G_{R_1 R_2 \cdots R_m}(x_1, x_2, \cdots, x_m) \rangle$ 为复数域上多元多项式环 $C[x_1, x_2, \cdots, x_m]$ 的由 $G_{R_1 R_2 \cdots R_m}(x_1, x_2, \cdots, x_m)$ 生成的理想值.

因为 $(d_t \omega_t^{i_t})^{n_t} - \prod_{j_t=1}^{n_t} d_{t,j_t} = 0$, 且 $u(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) =$

$u'(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}), 0 \leq i_t \leq n_t - 1, t = 1, \cdots, m$. 所以,

$$u(R_1, \cdots, R_m) = u'(R_1, \cdots, R_m).$$

又由于,

$$B = u(R_1, R_2, \cdots, R_m) = (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) \text{diag}(u(d_1, \cdots, d_m), \cdots, u(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}), \cdots, u(d_1 \omega_1^{n_1-1}, \cdots, d_m \omega_m^{n_m-1})) (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t)^{-1} = (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) \cdot \text{diag}(\frac{1}{f(d_1, \cdots, d_m)}, \cdots, \frac{1}{f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})}, \cdots, \frac{1}{f(d_1 \omega_1^{n_1-1}, \cdots, d_m \omega_m^{n_m-1})}) (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t)^{-1},$$

因此, $BA = I$.

由于 $B = u(R_1, \cdots, R_m)$ 是 $f(R_1, \cdots, R_m)$ 的逆, 则多项式 $u'(x)$ 可用 Euclid 算法求得.

为简化计算, 设 $f(x_1, \cdots, x_m)$ 系数是 a 且 $a \neq 0$, 令 $f'(x_1, \cdots, x_m) = f(x_1, \cdots, x_m) / a$, 则 $f(x_1, \cdots, x_m) = af'(x_1, \cdots, x_m)$, $f'(x_1, \cdots, x_m)$ 的首项系数是 1.

定理 4 设 $A = SCACIRC_{R_1 \cdots R_m}^{n_1 \cdots n_m}(a_{0, \cdots, 0}, \cdots, a_{j_1, \cdots, j_m}, \cdots, a_{n_1-1, \cdots, n_m-1}) \in SC_{R_1, \cdots, R_m}^{n_1, \cdots, n_m}$ 是一个奇异的 m 重对角因子循环矩阵, A 的表示多项式为 $f(x_1, \cdots, x_m)$, 假设 A 有 k 个非零特征值, 不失一般性, 可设 $g_{R_1}(x), \cdots, g_{R_m}(x)$ 的 $k, k+1, \cdots, n_1 n_2 \cdots n_m - 1$ 个根组 $(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})$ 使得 $f(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) = 0$, 并且 $d_t, d_t \omega_t, \cdots, d_t \omega_t^{n_t-1}$ 是 $g_{R_t}(x) = x^{n_t} - \prod_{j_t=1}^{n_t} d_{t,j_t}$, $t = 1, \cdots, m$ 的

根. 令 $g_1(x_1, x_2, \cdots, x_m) = \prod_{t=1}^m (x_t - d_t \omega_t^{i_t})$, 其中 $(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})$ 是 $g_{R_1}(x), \cdots, g_{R_m}(x)$ 的 $0, 1, \cdots, k-1$ 个根组,

令 $g_2(x_1, \cdots, x_m) = \prod_{t=1}^m (x_t - d_t \omega_t^{i_t})$, 其中 $(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m})$ 是 $g_{R_1}(x), \cdots, g_{R_m}(x)$ 的 $k, k+1, \cdots, n_1 n_2 \cdots n_m - 1$ 个根组, 且 $f_1(x_1, \cdots, x_m) = f(x_1, \cdots, x_m) g_2(x_1, \cdots, x_m)$, 则存在 1 个多项式 $u_1(x_1, \cdots, x_m)$ 满足:

$$u_1(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}) = 1 / f_1(d_1 \omega_1^{i_1}, \cdots, d_m \omega_m^{i_m}),$$

令 $u(x_1, \cdots, x_m) = u_1(x_1, \cdots, x_m) g_2(x_1, \cdots, x_m)$, 则:

$$A^+ = u(R_1, R_2, \cdots, R_m).$$

证明 由于,

$$x^{n_t} - \prod_{j_t=1}^{n_t} d_{t,j_t} = \prod_{i_t=0}^{n_t-1} (x - d_t \omega_t^{i_t}), t = 1, \cdots, m,$$

由 $g_1(x_1, \cdots, x_m), g_2(x_1, \cdots, x_m)$ 定义可得:

$$(g_1(x_1, \dots, x_m), g_2(x_1, \dots, x_m)) = 1.$$

又由条件可知:

$$(g_1(x_1, \dots, x_m), f(x_1, \dots, x_m)) = 1,$$

于是, $(g_1(x_1, \dots, x_m), f_1(x_1, \dots, x_m)) = 1.$

因此, 存在 $u_2(x_1, \dots, x_m), v(x_1, \dots, x_m)$ 满足:

$$u_2(x_1, \dots, x_m)f_1(x_1, \dots, x_m) + \\ v(x_1, \dots, x_m)g_1(x_1, \dots, x_m) = 1.$$

当 $(x_1, x_2, \dots, x_m) = (d_1\omega_1^i, \dots, d_m\omega_m^i)$ 是 $g_{R_i}(x), \dots, g_{R_m}(x)$ 的 $0, 1, \dots, k-1$ 个根组时, $g_1(d_1\omega_1^i, \dots, d_m\omega_m^i) = 0$, 所以, $u_2(d_1\omega_1^i, \dots, d_m\omega_m^i)f_1(d_1\omega_1^i, \dots, d_m\omega_m^i) = 1.$

令 $u_1(x_1, \dots, x_m) - u_2(x_1, \dots, x_m) \in \langle g_1(x_1, \dots, x_m) \rangle$, 所以该定理中 $u_1(x_1, \dots, x_m)$ 的存在性得以证明, 因此, $u(x_1, \dots, x_m) = u_1(x_1, \dots, x_m)g_2(x_1, \dots, x_m).$

当 $(d_1\omega_1^i, \dots, d_m\omega_m^i)$ 是 $g_{R_i}(x), \dots, g_{R_m}(x)$ 的 $k+1, \dots, n_1n_2 \dots n_m - 1$ 个根组时,

$$u(d_1\omega_1^i, \dots, d_m\omega_m^i) = 0.$$

当 $(d_1\omega_1^i, \dots, d_m\omega_m^i)$ 是 $g_{R_i}(x), \dots, g_{R_m}(x)$ 的 $0, 1, \dots, k-1$ 个根组时,

$$u(d_1\omega_1^i, \dots, d_m\omega_m^i) = u_1(d_1\omega_1^i, \dots, d_m\omega_m^i) \cdot \\ g_2(d_1\omega_1^i, \dots, d_m\omega_m^i) = g_2(d_1\omega_1^i, \dots, d_m\omega_m^i) / \\ f_1(d_1\omega_1^i, \dots, d_m\omega_m^i) = 1 / f(d_1\omega_1^i, \dots, d_m\omega_m^i),$$

且,

$$B = u(R_1, R_2, \dots, R_m) = (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) \text{diag}(u(d_1, \dots, \\ d_m), \dots, u(d_1\omega_1^i, \dots, d_m\omega_m^i), \dots, u(d_1\omega_1^{n_1-1}, \dots, \\ d_m\omega_m^{n_m-1})) (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t)^{-1} = (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t) \cdot \\ \text{diag}(\frac{1}{f(d_1, \dots, d_m)}, \frac{1}{f(d_1\omega_1, \dots, d_m\omega_m)}, \dots, \\ \frac{1}{f(d_1\omega_1^{k_1-1}, \dots, d_m\omega_m^{k_m-1})}, 0, \dots, 0) (\bigotimes_{t=1}^m \Delta_t \bigotimes_{t=1}^m F_t)^{-1}.$$

由定理 1 可知, $B = u(R_1, R_2, \dots, R_m) = A^+.$

2 m 重对角因子循环矩阵求逆和广义逆的算法

综前所述, 可得到 m 重对角因子循环矩阵的逆、群逆和 Moore-Penrose 逆的算法如下.

步骤 1 由 $A = SCACIRC_{R_1 \dots R_m}^{n_1 \dots n_m}(a_{0, \dots, 0}, \dots, a_{j_1, \dots, j_m}, \dots, a_{n_1-1, \dots, n_m-1})$ 得到:

$$f(x_1, \dots, x_m) = \\ \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \dots \sum_{j_m=0}^{n_m-1} a_{j_1, \dots, j_m} \prod_{t=1}^m (\prod_{s=0}^{j_t} d_{t,s})^{-1} \prod_{t=1}^m x_t^{j_t},$$

$$x^{n_t} - \prod_{j_t=1}^{n_t} d_{t,j_t},$$

计算出 $G_{R_1 R_2 \dots R_m}(x_1, x_2, \dots, x_m)$. 令 $r_{-1}(x_1, x_2, \dots, x_m) = G_{R_1 R_2 \dots R_m}(x_1, x_2, \dots, x_m)$, $r_0(x_1, \dots, x_m) = f(x_1, \dots, x_m)$, $u_{-1}(x_1, \dots, x_m) = 0$, $u_0(x_1, \dots, x_m) = 1$, $t = 1, \dots, m$.

步骤 2 利用多元多项式的除法, 由下式:

$$r_{i+1}(x_1, \dots, x_m) = r_{i-1}(x_1, \dots, x_m) - \\ q_i(x_1, \dots, x_m)r_i(x_1, \dots, x_m),$$

算出 $q_i(x_1, \dots, x_m)$, $r_{i+1}(x_1, \dots, x_m)$, $i = 0, 1, \dots, m$.

步骤 3 若 $r_{i+1}(x_1, \dots, x_m)$ 的首相系数是 c_{i+1} , 令:

$$r_{i+1}(x_1, \dots, x_m) \leftarrow r_{i+1}(x_1, \dots, x_m) / c_{i+1}, i = 0, 1, \dots, m.$$

步骤 4 由下式:

$$u_{i+1}(x_1, \dots, x_m) = (u_{i-1}(x_1, \dots, x_m) - \\ q_i(x_1, \dots, x_m)u_i(x_1, \dots, x_m)) / c_{i+1},$$

算出 $u_{i+1}(x_1, \dots, x_m)$, 转步骤 2;

步骤 5 如果 $r_k(x_1, \dots, x_m) = 1$, $u_k(x_1, \dots, x_m)$ 是 A^{-1} 的表示多项式, 则 $A^{-1} = u_k(R_1, R_2, \dots, R_m)$; 否则, $r_k(x_1, \dots, x_m) = 0$, 则 $r_{k-1}(x_1, \dots, x_m)$ 是 $f(x_1, \dots, x_m)$ 和 $G_{R_1 R_2 \dots R_m}(x_1, x_2, \dots, x_m)$ 最大公因式. 令:

$$r(x_1, x_2, \dots, x_m) = r_{k-1}(x_1, x_2, \dots, x_m), \\ r_{-1}(x_1, x_2, \dots, x_m),$$

满足:

$$r_{-1}(x_1, x_2, \dots, x_m)r_{k-1}(x_1, x_2, \dots, x_m) = \\ G_{R_1 R_2 \dots R_m}(x_1, x_2, \dots, x_m), \\ \{r_0(x_1, x_2, \dots, x_m) - f(x_1, x_2, \dots, x_m)r_{k-1}(x_1, \\ x_2, \dots, x_m)\} \in \langle r_{-1}(x_1, x_2, \dots, x_m) \rangle, \\ u_{-1}(x_1, x_2, \dots, x_m) = 0, u_0(x_1, x_2, \dots, x_m) = 1,$$

转步骤 2;

步骤 6 一直计算, 直到出现 $r_k(x_1, \dots, x_m) = 1$, 则 $u(x_1, \dots, x_m)$ 满足 $\{u(x_1, \dots, x_m) - u_k(x_1, \dots, x_m)r(x_1, \dots, x_m)\} \in \langle G_{R_1 R_2 \dots R_m}(x_1, \dots, x_m) \rangle$ 是 $A^\#$ 的表示多项式, 因此 $A^\# = u(R_1, R_2, \dots, R_m)$. 进一步可知, 如果 R_i 是正规的, 则 $A^+ = A^\# = u(R_1, R_2, \dots, R_m)$.

3 算法工作量

如果 m 重对角因子循环矩阵是非奇异的, 工作量可分为两部分. 假设 m 重对角因子循环矩阵 A 的阶数是 $n_1 n_2 \dots n_m$, 则 $\deg(f(x)) = n_1 n_2 \dots n_m - m$, 首先多元多项式除法的工作量为:

$$r_{i+1}(x_1, \dots, x_m) = r_{i-1}(x_1, \dots, x_m) - \\ q_i(x_1, \dots, x_m)r_i(x_1, \dots, x_m),$$

$$r_{i+1}(x_1, \dots, x_m) \leftarrow r_{i+1}(x_1, \dots, x_m) / c_{i+1}, i = 0, 1, \dots, m.$$

显然, 多元多项式除法不超过 $n_1 n_2 \cdots n_m - m$ 次. 如果只计算 $n_1 n_2 \cdots n_m - m$ 次, 则 $\deg(q_i(x_1, \dots, x_m)) = m$, $q_i(x_1, \dots, x_m), r_i(x_1, \dots, x_m)$ 首项系数都是 1, 于是多元多项式除法工作量为 $2 \sum_{i_1=1}^{n_1-1} \sum_{i_2=1}^{n_2-1} \cdots \sum_{i_m=1}^{n_m-1} i_1 i_2 \cdots i_m$, 不超过 $2n_1 n_2 \cdots n_m$, 如果计算少于 $n_1 n_2 \cdots n_m - m$ 次, 显然 $\deg(q_i(x_1, \dots, x_m)) > m$, 则需要简化多项式除法的次数. 此部分工作量不超过 $2n_1 n_2 \cdots n_m$.

其次, 讨论多项式乘法工作量为:

$$u_{i+1}(x_1, \dots, x_m) = (u_{i+1}(x_1, \dots, x_m) - q_i(x_1, \dots, x_m) \cdot u_i(x_1, \dots, x_m)) / c_{i+1}, i = 0, 1, \dots.$$

由算法可知, 多项式乘法不超过 $n_1 n_2 \cdots n_m - m$ 次. 如果计算 $n_1 n_2 \cdots n_m - m$ 次, 则 $\deg(q_i(x_1, \dots, x_m)) = m$, $q_i(x_1, \dots, x_m), r_i(x_1, \dots, x_m)$ 首项系数都是 1, 则多元多项式除法工作量为 $2 \sum_{i_1=1}^{n_1-1} \sum_{i_2=1}^{n_2-1} \cdots \sum_{i_m=1}^{n_m-1} i_1 i_2 \cdots i_m$, 不超过 $2n_1 n_2 \cdots n_m$, 如果计算少于 $n_1 n_2 \cdots n_m - m$ 次, 该部分乘法工作量不超过 $2n_1 n_2 \cdots n_m$, 因此总工作量不超过 $4n_1 n_2 \cdots n_m$.

当 m 重对角因子循环矩阵是奇异的时, 算法计算群逆或者 Moore-Penrose 逆的总工作量不超过 $8n_1 n_2 \cdots n_m$.

4 算法可行性

参考文献[5-6]介绍了 Mathematica、Maple 等软件没有实现的多元多项式除法运算, 而利用设计的存储结构及多项式的运算能非常容易实现, 运算速度非常快. 笔者在此实现了多元多项式的运算, 再利用第三部分算法, 也就实现了 m 重对角因子循环矩阵求逆和广义逆的算法.

参考文献:

- [1] 江兆林. 广义循环矩阵及其应用[D]. 西安: 西安电子科技大学, 2003.
- [2] 江兆林, 刘三阳. 置换因子循环矩阵求逆和广义逆的 Euclid 算法[J]. 西安电子科技大学学报: 自然科学版, 2004, 31(1):148-152.
- [3] 江兆林, 叶留青, 高淑萍. 友循环矩阵求逆和广义逆的 Euclid 算法[J]. 工程数学学报, 2004, 21(2):227-232.
- [4] Jiang Zhaolin, Ye Liuqing, Deng Fang'an. The explicit expressions of level $-k(r_1, r_2, \dots, r_k)$ circulant matrices and their some properties[J]. Chinese Quarterly Journal of Mathematics, 1998, 13(2):91-96.
- [5] 方文波. 可视化符号运算系统 VSCS 的设计与实现[J]. 计算机工程, 2003, 29(6):183-185.
- [6] 方文波. 多元多项式的存储结构及四则运算的实现[J]. 纺织高校基础科学学报, 2003, 19(3):257-260.

Euclid Algorithm for Finding Inverse and Generalized Inverse of m Multiplicity Diagonal Factor Circulant Matrix

HOU Dong-jie, CEN Jian-miao*

(Faculty of Science, Ningbo University, Ningbo 315211, China)

Abstract: A new algorithm for finding the inverse of a nonsingular m multiplicity diagonal factor circulant matrix is presented using the Euclid algorithm of polynomial. The presented algorithm is extended to compute the Moore-Penrose inverse of a singular m multiplicity diagonal factor circulant matrix. The steps in derivation are given in detail.

Key words: m multiplicity diagonally factor circulant matrix; inverse; group inverse; Moore-Penrose inverse; Euclid algorithm

(责任编辑 章践立)