

Pfaffianization of Generalized Variable-coefficient KP Equation

WU Ling, YU Ya-xuan*

(Faculty of Science, Ningbo University, Ningbo 315211, China)

Abstract: The generalized coupled variable-coefficient KP system is constructed using the pfaffianization procedure. Furthermore, the Wronski-type pfaffian solution and the Gramm-type pfaffian solution are presented for the pfaffianized KP equation with the partial application of pfaffian technique.

Key words: generalized variable-coefficient KP equation; Pfaffianization; Wronski-type Pfaffian solution; Gramm-type Pfaffian solution

CLC number: O340

Document code: A

Article ID: 1001-5132 (2015) 04-0100-04

Mathematicians and physicists use the soliton equations to characterize different kinds of nonlinear complex phenomena in some branches of modern science, such as the plasma physics, biology and fiber optics, fluid dynamics, Bose-Einstein condensations, solid state physics^[1-5]. Variable-coefficients KP equations can be used to describe applications of a wide range of physics more actual than the constant coefficients equations.

In the early 1990s, a procedure for generalizing equations from the KP hierarchy were developed by Ohta and Hirota to produce coupled system of equations with solutions in the form of pfaffians^[6-7]. In last several years, KP equations have taken more attention in producing coupled systems by applying the pfaffianization, such as the non-isospectral and variable-coefficient KP equation^[8], generalized variable-coefficient KP equation^[9-12], non-isospectral KP equation^[13], non-isospectral modified KP equation^[14], the discrete KP equation^[15], (3+1)-dimensional generalized BKP equation^[16], (3+1)-dimensional generalized KP equation^[17] and so on.

The author puts more attention to study the

following variable-coefficient KP equation:

$$[u_t + f(t)uu_x + g(t)u_{xxx} + l(t)u + q(t)u_x + n(t)u_y]_x + m(t)u_{yy} = 0, \quad (1)$$

where $f(t) \neq 0$, $g(t) \neq 0$, $l(t)$, $q(t) \neq 0$, $m(t)$ and $n(t)$ are all arbitrary functions of t . Eq.(1) can become many famous nonlinear wave equations, for example the standard KP equation and cylindrical KP equation, where $f(t)$, $g(t)$, $l(t)$, $q(t)$, $m(t)$ and $n(t)$ have special values. Grammian solutions^[18], analytic multi-soliton solutions, auto-Backlund transformation in bilinear form and Lax pair^[19] have been studied for the Eq.(1).

1 Pfaffianization and Wronski-type Pfaffian solution

Through the dependent variable transformation:

$$u(x, y, t) = \frac{12}{c_0} e^{-\int l(t) dt} [\ln F(x, y, t)]_{xx}, \quad (2)$$

with $6/c_0 = \lambda$ is an arbitrary non-zero constant. Eq.(1) can be mapped into the variable-coefficient bilinear equation^[18],

$$[D_x D_t + g(t)D_x^4 + q(t)D_x^2 + m(t)D_y^2 +$$

Received date: 2014-06-30.

JOURNAL OF NINGBO UNIVERSITY (NSEE): <http://journal.g.nbu.edu.cn/>

Foundation items: Supported by the National Natural Science Foundation of China (11271211, 11271210); K.C.Wong Magna Fund in Ningbo University.

The first author: WU Ling (1988-), female, Xinyang Henan, postgraduate, research domain: soliton system. E-mail: wuling1988226@163.com

***Corresponding author:** YU Ya-xuan (1979-), female, Neimenggu Tongliao, associate professor, research domain: soliton and integrable system. E-mail: yuyaxuan@nbu.edu.cn

$$n(t)D_x D_y]F \cdot F = 0, \quad (3)$$

where D_x, D_y and D_t are the Hirota bilinear operator^[20].

We will employ the pfaffianization procedure to pfaffianize Eq.(3). We consider the pfaffian

$$F = (1, 2, \dots, 2N) = (\widehat{2N}), \quad (4)$$

whose entries satisfy the differential rule:

$$\begin{aligned} \partial(i, j) / \partial x &= (i+1, j) + (i, j+1), \\ \partial(i, j) / \partial y &= -3g(t)[(i+2, j) + (i, j+2)], \\ \partial(i, j) / \partial t &= -4g(t)[(i+3, j) + (i, j+3)] + 3g(t)n(t) \cdot \\ &\quad [(i+2, j) + (i, j+2)] - q(t)[(i+1, j) + (i, j+1)]. \end{aligned} \quad (5)$$

The entries of one special kind of the Wronski-type Pfaffian solution are expressed as:

$$(i, j) = \sum_{k=1}^M [\Phi_k^{(i)} \Psi_k^{(j)} - \Phi_k^{(j)} \Psi_k^{(i)}],$$

where M is an arbitrary natural number, and $\partial \Phi_k /$

$\partial x_n = \Phi_k^{(n)}, \partial \Psi_k / \partial x_n = \Psi_k^{(n)}, \Phi_k$ and Ψ_k satisfy

$$\begin{aligned} \Phi_{k,y} &= -3g(t)\Phi_{k,xx} \Psi_{k,y} = -3g(t)\Psi_{k,xx}, \\ \Phi_{k,t} &= -4g(t)\Phi_{k,xxx} + 3g(t)n(t)\Phi_{k,xx} - q(t)\Phi_{k,x}, \\ \Psi_{k,t} &= -4g(t)\Psi_{k,xxx} + 3g(t)n(t)\Psi_{k,xx} - q(t)\Psi_{k,x}. \end{aligned}$$

Under the condition (5), we have the differential rules for the Wronski-type Pfaffian:

$$\begin{aligned} \frac{\partial}{\partial x}(i_1, i_2, \dots, i_{2N}) &= \sum_{k=1}^{2N} (i_1, i_2, \dots, i_k + 1, \dots, i_{2N}), \\ \frac{\partial}{\partial y}(i_1, i_2, \dots, i_{2N}) &= \sum_{k=1}^{2N} [-3g(t)](i_1, i_2, \dots, i_k + 2, \dots, i_{2N}), \\ \frac{\partial}{\partial t}(i_1, i_2, \dots, i_{2N}) &= \sum_{k=1}^{2N} [-4g(t)(i_1, i_2, \dots, i_k + 3, \dots, i_{2N}) + \\ &\quad 3g(t)n(t)(i_1, i_2, \dots, i_k + 2, \dots, i_{2N}) - \\ &\quad q(t)(i_1, i_2, \dots, i_k + 1, \dots, i_{2N})], \end{aligned} \quad (6)$$

which are exactly the same as the differential rules for determinants^[20]. By means of these rules, various derivatives of the pfaffian F can be computed as:

$$\begin{aligned} F_x &= (\widehat{2N-1, 2N+1}), \\ F_{xx} &= (\widehat{2N-2, 2N, 2N+1}) + (\widehat{2N-1, 2N+2}), \\ F_{xxx} &= (\widehat{2N-3, 2N-1, 2N, 2N+1}) + 2(\widehat{2N-2, 2N, 2N+2}) + (\widehat{2N-1, 2N+3}), \\ F_{xxxx} &= (\widehat{2N-4, 2N-2, 2N-1, 2N, 2N+1}) + \\ &\quad (\widehat{2N-1, 2N+4}) + 3(\widehat{2N-3, 2N-1, 2N, 2N+2}) + 3(\widehat{2N-2, 2N, 2N+3}) + \\ &\quad 2(\widehat{2N-2, 2N+1, 2N+2}), \\ F_y &= -3g(t)[(\widehat{2N-1, 2N+2}) - (\widehat{2N-2, 2N, 2N+1})], \\ F_{xy} &= -3g(t)[(\widehat{2N-1, 2N+3}) - (\widehat{2N-3, 2N-1, 2N, 2N+1})], \end{aligned}$$

$$\begin{aligned} &2N, 2N+1)], \\ F_{yy} &= 9g(t)^2[(\widehat{2N-1, 2N+4}) + 2(\widehat{2N-2, 2N+1, 2N+2}) - (\widehat{2N-3, 2N-1, 2N, 2N+2}) - \\ &\quad (\widehat{2N-2, 2N, 2N+3}) + (\widehat{2N-4, 2N-2, 2N-1, 2N, 2N+1})], \\ F_t &= -4g(t)[(\widehat{2N-1, 2N+3}) - (\widehat{2N-2, 2N, 2N+2}) + (\widehat{2N-3, 2N-1, 2N, 2N+1})] + 3g(t)n(t) \cdot \\ &\quad [(\widehat{2N-1, 2N+2}) - (\widehat{2N-2, 2N, 2N+1})] - q(t)(\widehat{2N-1, 2N+1}), \\ F_{xt} &= -4g(t)[(\widehat{2N-1, 2N+4}) - (\widehat{2N-2, 2N+1, 2N+2}) + (\widehat{2N-4, 2N-2, 2N-1, 2N, 2N+1})] + 3g(t)n(t) \cdot \\ &\quad [(\widehat{2N-1, 2N+3}) - (\widehat{2N-3, 2N-1, 2N, 2N+1})] - q(t)[(\widehat{2N-2, 2N, 2N+1}) + (\widehat{2N-1, 2N+2})]. \end{aligned}$$

Substituting the above derivatives and F into the left of Eq.(3), we have

$$\begin{aligned} [D_x D_t + g(t)D_x^4 + q(t)D_x^2 + m(t)D_y^2 + n(t)D_x D_y]F \cdot \\ F = 24g(t)[(\widehat{2N})(\widehat{2N-2, 2N+1, 2N+2}) - \\ (\widehat{2N-1, 2N+1})(\widehat{2N-2, 2N, 2N+2}) + \\ (\widehat{2N-1, 2N+2})(\widehat{2N-2, 2N, 2N+1})]. \end{aligned} \quad (7)$$

While the Eq.(7) no longer satisfy the Eq.(3), following the Pfaffianization procedure, we introduce

$$\begin{aligned} G &= (1, 2, \dots, 2N+2) = (\widehat{2N+2}), \\ H &= (1, 2, \dots, 2N-2) = (\widehat{2N-2}), \end{aligned} \quad (8)$$

with N even, F, G, H satisfy the bilinear equations:

$$[D_x D_t + g(t)D_x^4 + q(t)D_x^2 + m(t)D_y^2 + n(t)D_x D_y]F \cdot F = 24g(t)GH, \quad (9)$$

$$\begin{aligned} [D_x - 2g(t)D_x^3 + n(t)D_y + q(t)D_x + aD_x D_y]G \cdot \\ F = 0, \end{aligned} \quad (10)$$

$$\begin{aligned} [D_x - 2g(t)D_x^3 + n(t)D_y + q(t)D_x - aD_x D_y]H \cdot \\ F = 0, \end{aligned} \quad (11)$$

with a is an arbitrary non-zero number.

We calculate the derivatives of G and H through the Eq.(5). Substitute F, G, H and the derivatives into Eq.(10) and (11), and they can be written as:

$$\begin{aligned} -12g(t)[(\widehat{2N})(\widehat{2N-1, 2N+1, 2N+2, 2N+3}) - \\ (\widehat{2N-1, 2N+1})(\widehat{2N, 2N+2, 2N+3}) + (\widehat{2N-1, 2N+2})(\widehat{2N+1, 2N+3}) - (\widehat{2N-1, 2N+3}) \cdot \\ (\widehat{2N+2})] = 0, \\ 12g(t)[(\widehat{2N-3, 2N+1})(\widehat{2N}) - (\widehat{2N-3, 2N}) \cdot \\ (\widehat{2N-1, 2N+1}) + (\widehat{2N-3, 2N-1})(\widehat{2N-2, 2N+1})] = 0, \end{aligned}$$

$$2N, 2N+1) - \widehat{(2N-2)(2N-3, 2N-1, 2N, 2N+1)} = 0,$$

which is nothing but the pfaffian identity.

Thus Eqs.(9)-(11) comprise the coupled variable-coefficient KP equations, and they have the Wronski-type pfaffian solution F, G and H with elements satisfy the condition (5).

Under the dependent variable transformations,

$$u(x, y, t) = \frac{12}{c_0} e^{-\int l(t) dt} [\ln F(x, y, t)]_{xx},$$

$$v = G/F, w = H/F,$$

the coupled systems (9)-(11) can be mapped into the following nonlinear differential equations,

$$\begin{aligned} & [u_t + f(t)uu_x + g(t)u_{xxx} + l(t)u + q(t)u_x + n(t)u_y]_x + \\ & m(t)u_{yy} - 24\lambda g(t)(vw)_{xx} = 0, \\ & v_t - 2g(t)v_{xxx} + n(t)v_y + [q(t) - f(t)u]v_x + \\ & a(v_{xy} + (v/\lambda) \int u_y dx) = 0, \\ & w_t - 2g(t)w_{xxx} + n(t)w_y + [q(t) - f(t)u]w_x - \\ & a(w_{xy} + (w/\lambda) \int u_y dx) = 0, \end{aligned} \quad (12)$$

which reduces to the variable-coefficient KP Eq.(1) as $v = w = 0$.

2 the Gramm-type Pfaffian solution

In this section, we discuss the Gramm-type pfaffian solution for the pfaffianized variable-coefficient KP equations (9)-(11). The Gramm-type pfaffian solution can be proposed as:

$$\begin{aligned} F &= (1, 2, \dots, 2N), \quad G = (d_0, d_1, 1, 2, \dots, 2N), \\ H &= (c_1, c_0, 1, 2, \dots, 2N), \end{aligned} \quad (13)$$

with the Pfaffian entries:

$$\begin{aligned} (i, j) &= c_{ij} + \int^x (\alpha_i \beta_j - \alpha_j \beta_i) dx, \quad c_{ji} = -c_{ji}, \\ (d_n, i) &= \partial^n \alpha_i / \partial x^n, \quad (c_m, j) = \partial^m \beta_j / \partial x^m, \\ (d_m, d_n) &= (c_m, c_n) = (d_m, c_n) = 0, \end{aligned} \quad (14)$$

where α_i and β_j satisfy the differential relations:

$$\begin{aligned} \alpha_{i,y} &= -3g(t)\alpha_{i,xx}, \quad \beta_{j,y} = 3g(t)\beta_{j,xx}, \\ \alpha_{i,t} &= -4g(t)\alpha_{i,xxx} + 3g(t)n(t)\alpha_{i,xx} - q(t)\alpha_{i,x}, \\ \beta_{j,t} &= -4g(t)\beta_{j,xxx} + 3g(t)n(t)\beta_{j,xx} - q(t)\beta_{j,x}. \end{aligned} \quad (15)$$

Based of the above differential rules, we obtain

$$\begin{aligned} \partial(i, j) / \partial x &= (c_0, d_0, i, j), \\ \partial(i, j) / \partial y &= -3g(t)[(c_0, d_1, i, j) - (c_1, d_0, i, j)], \end{aligned}$$

$$\begin{aligned} \partial(i, j) / \partial t &= -4g(t)[(c_0, d_2, i, j) - (c_1, d_1, i, j) + \\ & (c_2, d_0, i, j)] + 3g(t)n(t)[(c_0, d_1, i, j) - \\ & (c_1, d_0, i, j)] - q(t)(c_0, d_0, i, j). \end{aligned}$$

Various derivatives of F (for short $F = (\cdot)$) can be yielded as follows:

$$\begin{aligned} F_x &= (c_0, d_0, \cdot), \\ F_{xx} &= (c_0, d_1, \cdot) + (c_1, d_0, \cdot), \\ F_{xxx} &= (c_0, d_2, \cdot) + 2(c_1, d_1, \cdot) + (c_2, d_0, \cdot), \\ F_{xxxx} &= (c_0, d_3, \cdot) + 3(c_2, d_1, \cdot) + 2(c_0, d_0, c_1, d_1, \cdot) + \\ & 3(c_1, d_2, \cdot) + (c_3, d_0, \cdot), \\ F_y &= -3g(t)[(c_0, d_1, \cdot) - (c_1, d_0, \cdot)], \\ F_{xy} &= -3g(t)[(c_0, d_2, \cdot) - (c_2, d_0, \cdot)], \\ F_{yy} &= 9g(t)^2[(c_3, d_0, \cdot) - (c_2, d_1, \cdot) - (c_1, d_2, \cdot) + \\ & 2(c_0, d_0, c_1, d_1, \cdot) + (c_0, d_3, \cdot)], \\ F_t &= -4g(t)[(c_0, d_2, \cdot) - (c_1, d_1, \cdot) + (c_2, d_0, \cdot)] + \\ & 3g(t)n(t)[(c_0, d_1, \cdot) - (c_1, d_0, \cdot)] - q(t)(c_0, d_0, \cdot), \\ F_{xt} &= -4g(t)[(c_0, d_3, \cdot) - (c_0, d_0, c_1, d_1, \cdot) + (c_3, d_0, \cdot)] + \\ & 3g(t)n(t)[(c_0, d_2, \cdot) - (c_2, d_0, \cdot)] - q(t)[(c_0, \\ & d_1, \cdot) + (c_1, d_0, \cdot)]. \end{aligned}$$

We substitute the above derivatives, G and H into Eq.(8) and get the pfaffian identity:

$$\begin{aligned} 24g(t)[(c_0, d_0, c_1, d_1, \cdot)(\cdot) - (c_0, d_0, \cdot)(c_1, d_1, \cdot) + (c_0, \\ c_1, \cdot)(d_0, d_1, \cdot) - (c_0, d_1, \cdot)(d_0, c_1, \cdot)] = 0. \end{aligned}$$

Eq.(10) and (11) can also be reduced to pfaffian identities in the same way. Hereby, we have shown the Gramm-type pfaffian solution given by Eq.(13) satisfy the coupled systems (9)-(11).

3 conclusions

The coupled variable-coefficient KP equation have been presented through the pfaffianization procedure. Not only the Wronski-type pfaffian solution but also the Gramm-type pfaffian solution has been shown for the pfaffianized coupled variable-coefficient, which has been proved by the pfaffian identities.

References:

- [1] Lamb J G L. Elements of soliton theory[M]. New York: Wiley Presses, 1980:203-209.
- [2] Ablowitz M J, Clarkson P A. Solitons, nonlinear

- evolution equations and inverse scattering[M]. Cambridge: Cambridge University Press, 1991:70-100.
- [3] Agrawal G P. Nonlinear fiber optics[M]. New York: Academic Press, 1995:195-211.
- [4] Gao Yitian, Tian Bo. Variable-coefficient unstable nonlinear Schrödinger equation modeling electron beam plasma: Auto-Bäcklund transformation, soliton-typed and other analytical solutions[J]. Phys Plasmas, 2001,8:67-73.
- [5] Gao Yitian, Tian Bo. A variable-coefficient unstable nonlinear Schrödinger model for the electron beam plasmas and Rayleigh-Taylor instability in nonuniform plasmas: Solutions and observable effects[J]. Phys Plasmas, 2003, 10:4306-4313.
- [6] Hirota R, Ohta Y. Hierarchies of coupled soliton equations[J]. J Phys Soc Jpn, 1991, 60:798-809.
- [7] Hirota R. Direct methods in soliton theory[M]. Tokyo: Iwanami Shoten, 1992:157-176.
- [8] Sun Yepeng, Tam H W. Grammian solutions and pfaffianization of a non-isospectral and variable-coefficient Kadomtsev-Petviashvili equation[J]. J Math Anal Appl, 2008, 343:810-817.
- [9] Meng Xianghua, Tian Bo, Zhang Haiqiang. Pfaffianization of the generalized variable-coefficient Kadomtsev-Petviashvili equation[J]. Appl Math Comput, 2010, 217:1300-1305.
- [10] Zhang Yuanyuan, Zheng Ying, Zhang Hongqing. Pfaffianization of the variable-coefficient KP equation[J]. Chaos Soliton and Fractals, 2009, 39:1000-1004.
- [11] Ma Wenxiu, Xia Tiecheng. Pfaffianized systems for a generalized Kadomtsev-Petviashvili equation[J]. Phys Scr, 2013, 87:550031-550037.
- [12] Zhang Qingfan, Fan Engui. Pfaffianization of the variable-coefficient Kadomtsev-Petviashvili equation[J]. Chinese Physics, 2007, 16:1505-1509.
- [13] Yu Guofu, Tam H W. On the nonisospectral Kadomtsev-Petviashvili equation[J]. J Phys A: Math Gen, 2006, 39: 3367-3373.
- [14] Sun Yepeng, Tam H W. Grammian solutions and Pfaffianization of the non-isospectral modified Kadomtsev-Petviashvili equation[J]. Phys. Lett. A, 2008, 372: 4412-4417.
- [15] Gilson C R, Nimmo J J C, Tsujimoto S. Pfaffianization of the discrete KP equation[J]. J Phys A: Math Gen, 2001, 34:10569-10575.
- [16] Asaad M G, Ma Wenxiu. Pfaffian solutions to a (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation and its modified counterpart[J]. Appl Math Comput, 2012, 218:5524-5542.
- [17] Meng Xianghua, Xu Xiaoge. Pfaffianization of a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation [J]. Appl Math Lett, 2013, 26:612-616.
- [18] Ye Lingya, Lv Yineng, Zhang Yi, Jin Huiping. Grammian Solutions to a Variable-coefficient KP Equation[J]. Chin Phys Lett, 2008, 25:357-358.
- [19] Liang Yueqian, Wei Guangmei, Li Xiaonan. Transformations and multi-solitonic solutions for a generalized variable-coefficient Kadomtsev-Petviashvili equation[J]. Comput Math Appl, 2011, 61:3268-3277.
- [20] Hirota R. Soliton theory of direct methods[M]. Cambridge: Cambridge University Press, 2004:107-150.

一个广义变系数 KP 方程的 Pfaffianization 化

吴 玲, 于亚璇*

(宁波大学 理学院, 浙江 宁波 315211)

摘要: 通过使用 Pfaffianization 化程序, 产生一个广义耦合的变系数 KP 方程组, 并且利用 pfaffian 技术给出了耦合变系数 KP 方程组的 Wronski 型 pfaffian 式解和 Gram 型 pfaffian 式解.

关键词: 广义变系数 KP 方程; Pfaffianization 化; Wronski 型 pfaffian 式解; Gram 型 pfaffian 式解

(责任编辑 章践立)