

Stochastic Differential Equations with a Fractional Brownian Motion

ZHANG Ling-hao¹, TAO Xiang-xing^{2*}

(1.Faculty of Science, Ningbo University, Ningbo 315211, China;

2.Faculty of Science, Zhejiang University of Science and Technology, Hangzhou 310023, China)

Abstract: In this paper we investigate the local and global existence and uniqueness of mild solutions to stochastic differential equations perturbed by a fractional Brownian motion $B_Q^H(t)$: $du = (Au + uu_x)dt + g(t)dB_Q^H(t)$ with Hurst parameter $H \in (1/2, 1)$.

Key words: stochastic differential equations; fractional Brownian motion; Hurst parameter

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Consider the following stochastic initial-boundary differential equation

$$du = (Au + uu_x)dt + g(t)dB_Q^H(t), \quad (1)$$

$$u(0, t) = u(1, t) = 0, \quad (2)$$

$$u(x, 0) = u_0(x), x \in [0, 1], \quad (3)$$

under suitable conditions on the operator A , the coefficient functions g , and the initial value u_0 . Here $B_Q^H(t)$ denotes an fBm with $H \in (1/2, 1)$.

The existence and regularity properties of the solution of the stochastic Burgers' equation with a standard Brownian motion have been obtained by several authors; see e.g. Da Prato et al^[1] and Gyongy and Nualart^[2]. In [3], the local and global existence and uniqueness results for the stochastic Burgers' equation driven by fBm is proved when $H > 1/4$, B^H is a cylindrical fBm and A be a Laplacian operator. In [4], the author prove existence, uniqueness and exponential asymptotic behavior of mild solutions to stochastic delay evolution equations with a fBm for $H > 1/2$.

The aim of this paper is to obtain the global

well-posedness of the solution for the eq. (1), with boundary and initial conditions eq. (2) and (3). In this paper we consider the case where $H > 1/2$, $B_Q^H(t)$ is a Q -cylindrical fractional Brownian motion. This case cannot be covered by the above previous results.

The paper is organized as follows. In section 2 some necessary preliminaries on the stochastic integration with respect to fBm are established. Also a technical lemma which is crucial in our stability analysis is proved. In section 3 we prove the regularity properties of the stochastic integral. In the last section, we prove local and global existence results to the eq. (1) to (3).

1 Preliminaries

In this section we introduce the fractional Brownian motion as well as the Wiener integral with respect to it. We also establish some important results which will be needed throughout the paper.

Let $(\beta^H(t), t \geq 0)$ be a fractional Brownian

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The first author: ZHANG Ling-hao (1986-), male, Xiantao Hubei, post of graduate student, research domain: stochastic differential equation.

E-mail: zhanglinghao@sina.com

***Corresponding author:** TAO Xiang-xing (1965-), male, Taizhou Zhejiang, doctor/professor, research domain: differential equation. E-mail: xxtau@163.com

motion with Hurst parameter $H \in (1/2, 1)$, defined in a complete probability space $(\Omega, \mathcal{F}, P)$. Namely, $(\beta^H(t), t \geq 0)$ is a mean zero Gaussian process with covariance

$$R_H(t, s) = E[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Now we aim at introducing the Wiener integral with respect to the one-dimensional fBm β^H . Let $T > 0$ and denote by Λ the linear space of R -value step functions on $[0, T]$, that is, $\phi \in \Lambda$ if

$$\phi(t) = \sum_{i=1}^{n-1} x_i 1_{[t_i, t_{i+1})}(t),$$

where $t \in [0, T]$, $x_i \in R$ and $0 = t_1 < t_2 < \dots < t_n = T$. For $\phi \in \Lambda$ we define its Wiener integral with respect to β^H as

$$\int_0^T \phi(s) d\beta^H(s) = \sum_{i=1}^{n-1} x_i (\beta^H(t_{i+1}) - \beta^H(t_i)).$$

Let H be the Hilbert space defined as the closure of Λ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_H = R_H(t, s).$$

Then the mapping

$$\phi = \sum_{i=1}^{n-1} x_i 1_{[t_i, t_{i+1})} \mapsto \int_0^T \phi(s) d\beta^H(s)$$

is an isometry between Λ and the linear space $\text{span}\{\beta^H, t \in [0, T]\}$, which can be extended to an isometry between H and the first Wiener chaos of the fBm $\overline{\text{span}}^{L^2(\Omega)}\{\beta^H, t \in [0, T]\}$ [5]. The image of an element $\phi \in H$ by this isometry the Wiener integral of ϕ with respect to β^H . Our next goal is to give an explicit expression of this integral. To this end, consider the kernel

$$K_H(t, s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du,$$

where $c_H = (H(2H-1)/B(2-2H, H-1/2))^{1/2}$, with B denoting the Beta function, and $t > s$. It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t, s) = c_H (t/s)^{H-1/2} (t-s)^{H-3/2}.$$

Consider the linear operator $K_H^*: \Lambda \mapsto L^2([0, T])$ given by

$$(K_H^* \phi)(s) = \int_s^t \phi(t) \frac{\partial K_H}{\partial t}(t, s) dt.$$

Then $(K_H^* 1_{[0,t]})(s) = K_H(t, s) 1_{[0,t]}(s)$.

We refer to [6] for the proof of the fact that K_H^* is an isometry which can be extended to H .

Considering $W = \{W(t), t \in [0, T]\}$ defined by

$$W(t) = \beta^H((K_H^*)^{-1} 1_{[0,t]}),$$

it turn out that W is a Wiener process and β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) dW(s).$$

In addition, for any $\phi \in H$,

$$\int_0^T \phi(s) d\beta^H(s) = \int_0^T (K_H^* \phi)(t) dW(t),$$

if and only if $K_H^* \phi \in L^2([0, T])$.

Also denoting $L_H^2([0, T]) = \{\phi \in H, K_H^* \phi \in L^2([0, T])\}$, since $H > 1/2$, we have [7]

$$L^{1/H}([0, T]) \subset L_H^2([0, T]). \quad (4)$$

Next we are interested in considering an fBm with values in a Hilbert spaces and giving the definition of the corresponding stochastic integral.

Let $(U, |\cdot|_U, (\cdot, \cdot)_U)$ and $(K, |\cdot|_K, (\cdot, \cdot)_K)$ be separable Hilbert spaces. Let $L(K, U)$ denote the space of all bounded linear operators from K to U . Let $Q \in L(K, K)$ be a non-negative self-adjoint operator. Denote by $L_Q^0(K, U)$ the space of all $\xi \in L(K, U)$ such that $\xi Q^{1/2}$ is a Hilbert-Schmidt operator. The norm is given by

$$|\xi|_{L_Q^0(K, U)}^2 = |\xi Q^{1/2}|_{HS}^2.$$

Then ξ is called a Q -Hilbert-Schmidt operator from K to U .

Let $\{\beta_n^H(t)\}_{n \in N}$ be a sequence of fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, P)$. When one considers the following series

$$\sum_{n=1}^{\infty} \beta_n^H(t) e_n,$$

where $\{e_n\}_{n \in N}$ is a complete orthonormal basis in $C([0, 1]; K)$, this series does not necessarily converge in the space K . We denote by $C([0, 1]; K)$ the space of all continuous functions from $[0, 1]$ into K . We consider a K -valued stochastic process $B_Q^H(t)$ given formally by the following series

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{1/2} e_n.$$

If Q is a non-negative self-adjoint trace class operator, then this series converges in the space K , that is, it holds that $B_Q^H(t) \in L^2(\Omega, K)$. Then, we say that the above $B_Q^H(t)$ is a K -valued Q -cylindrical fractional Brownian motion with covariance operator Q . For example, if $\{\sigma_n\}_{n \in \mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$, assuming that Q is a nuclear operator in K ($\sum_{n=1}^{\infty} \sigma_n < \infty$), then the stochastic process

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{1/2} e_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n} \beta_n^H(t) e_n,$$

is well-defined as a K -valued Q -cylindrical fractional Brownian motion.

Let $\varphi: [0, T] \rightarrow L_Q^0(K, U)$ such that

$$\sum_{n=1}^{\infty} \|K_H^*(\varphi Q^{1/2} e_n)\|_{L^2([0, T]; U)} < \infty. \quad (5)$$

Definition 1 Let $\varphi: [0, T] \rightarrow L_Q^0(K, U)$ satisfy (5). Then, its stochastic integral with respect to the fBm $B_Q^H(t)$ is defined, as follows

$$\int_0^t \varphi(s) dB_Q^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{1/2} e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t (K_H^*(\varphi Q^{1/2} e_n))(s) dW(s).$$

Notice that if

$$\sum_{n=1}^{\infty} \|\varphi Q^{1/2} e_n\|_{L^{1/H}([0, T]; U)} < \infty, \quad (6)$$

then in particular (5) holds, which follows immediately from (4).

The following useful result holds:

Lemma 1^[4] For any $\varphi: [0, T] \rightarrow L_Q^0(K, U)$ such that (6) holds, and for any $\alpha, \beta \in [0, T]$ with $\alpha > \beta$,

$$E \left| \int_{\beta}^{\alpha} \varphi(s) dB_Q^H(s) \right|_U^2 \leq cH(2H-1)(\alpha - \beta)^{2H-1}.$$

$$\sum_{n=1}^{\infty} \int_{\beta}^{\alpha} |\varphi(s) Q^{1/2} e_n|_U^2 ds,$$

where $c = c(H)$. If, in addition, $\sum_{n=1}^{\infty} \|\varphi(t) Q^{1/2} e_n\|_U$ is uniformly convergent for $t \in [0, T]$, then

$$E \left| \int_{\beta}^{\alpha} \varphi(s) dB_Q^H(s) \right|_U^2 \leq cH(2H-1)(\alpha - \beta)^{2H-1}.$$

$$\int_{\beta}^{\alpha} |\varphi(s)|_{L_Q^0(K, U)}^2 ds. \quad (7)$$

2 The stochastic integral

The linear stochastic differential equations corresponding to problem (1)–(3) is given by

$$du = Audt + g(t)dB_Q^H(t), \quad u(0) = u_0, \quad (8)$$

where $B_Q^H(t)$ is the fractional Brownian motion which was introduced in the previous section, $A: \text{Dom}(A) \subset U \rightarrow U$ is the infinitesimal generator of a strongly continuous semi-group $S(\cdot)$ on U , that is, for $t \geq 0$, it holds

$$\|S(t)\|_U \leq Me^{\rho t}, \quad M \geq 1, \quad \rho \in \mathbb{R},$$

$g: [0, T] \rightarrow L_Q^0(K, U)$, we assume the following conditions: For the complete orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ in K , we have

$$(i) \sum_{n=1}^{\infty} \|gQ^{1/2}e_n\|_{L^2([0, T]; U)} < \infty.$$

(ii) $\sum_{n=1}^{\infty} \|gQ^{1/2}e_n\|_U$ is uniformly convergent for $t \in [0, T]$.

(iii) $\sum_{n=1}^{\infty} \|gQ^{1/2}e_n(x)\|_U$ is uniformly convergent for $x \in [0, 1]$.

The mild form of the linear problem (8), if it exists, can be written as

$$u(t) = S(t)u_0 + \int_0^t S(t-s)g(s)dB_Q^H(s).$$

In the study of linear and nonlinear equations, just as the properties of the stochastic integral deduced by standard Gaussian process^[8], it is of great importance to establish the basic properties of the stochastic integral

$$Z(t) = \int_0^t S(t-s)g(s)dB_Q^H(s). \quad (9)$$

In this section we prove the continuous of the stochastic integral $Z(t)$. We state the basic properties of the stochastic integral as follow.

Theorem 1 The stochastic integral $Z: t \mapsto \int_0^t S(t-s)g(s)dB_Q^H(s)$ is well-defined. It is continuous with respect to $x \in [0, 1]$, and $t \geq 0$ for Hurst parameter $H \in (1/2, 1)$.

Proof We first prove that the stochastic integral is well defined.

Notice that, thanks to (i) and the fact that

$$H \in (1/2, 1),$$

$$\sum_{n=1}^{\infty} \|gQ^{1/2}e_n\|_{L^{1/H}([0,T];U)} < \infty$$

holds. Since

$$L^{1/H}([0,T]) \subset L^2_H([0,T]),$$

we have

$$\sum_{n=1}^{\infty} \|K_H^*(gQ^{1/2}e_n)\|_{L^2([0,T];U)} < \infty,$$

which implies that the stochastic integral $Z(t)$ is well-defined since $S(\cdot)$ is a strong continuous semigroup. Moreover, (i) together with (ii) immediately imply that, for every $t \in [0, T]$,

$$\int_0^t |g(s)|_{L^2_0(K,U)}^2 ds < \infty.$$

We check that the well-defined stochastic integral possesses the regularity. We now prove that for any positive $T > 0$, $Z(t)$ belongs to the space $C([0, T]; U)$. Consider $\sigma > 0$ small enough, we have

$$\begin{aligned} E \left| \int_0^{t+\sigma} S(t+\sigma-s)g(s)dB_Q^H(s) - \int_0^t S(t-s)g(s)dB_Q^H(s) \right|^2 &\leq \\ 2E \left| \int_0^t (S(t+\sigma-s) - S(t-s))g(s)dB_Q^H(s) \right|^2 + \\ 2E \left| \int_t^{t+\sigma} S(t+\sigma-s)g(s)dB_Q^H(s) \right|^2 &:= \\ J_1 + J_2. \end{aligned}$$

Applying inequality (7) to J_1 ,

$$\begin{aligned} J_1 &\leq 2cH(2H-1)t^{2H-1} \int_0^t |S(t-s) \cdot \\ &\quad (s(\sigma) - Id)g(s)|_{L^2_0(K,U)}^2 ds \leq \\ &\quad 2cH(2H-1)t^{2H-1}M^2e^{2\rho t} \cdot \\ &\quad \int_0^t |s(\sigma) - Id)g(s)|_{L^2_0(K,U)}^2 ds \rightarrow 0, \end{aligned}$$

when $\sigma \rightarrow 0$ thanks to the Lebesgue majorant Theorem, since, for every s fixed,

$$\begin{aligned} s(\sigma)g(s) &\rightarrow g(s), \\ |s(\sigma)g(s)|_{L^2_0(K,U)} &\leq Me^{\rho\sigma}|g(s)|_{L^2_0(K,U)}. \end{aligned}$$

Applying now (7) to J_2 we obtain

$$\begin{aligned} J_2 &\leq 2cH(2H-1)\sigma^{2H-1}M^2e^{2\rho\sigma} \cdot \\ &\quad \int_t^{t+\sigma} |g(s)|_{L^2_0(K,U)}^2 ds \rightarrow 0, \end{aligned}$$

when $\sigma \rightarrow 0$. Therefore the stochastic integral $Z(t)$ is continuous with respect to t .

For the continuous of Z with respect to x , we have

$$\begin{aligned} E \left| \int_0^t S(t-s)g(s)dB_Q^H(s) \right|^2 &\leq \\ cH(2H-1)t^{2H-1} \int_0^t |S(t-s)g(s)|_{L^2_0(K,U)}^2 ds &\leq \\ cH(2H-1)t^{2H-1}M^2e^{2\rho t} \int_0^t |g(s)|_{L^2_0(K,U)}^2 ds &\leq \\ cH(2H-1)t^{2H-1}M^2e^{2\rho t} \int_0^t |g(s)Q^{1/2}|_{HS}^2 ds &\leq \\ cH(2H-1)M^2e^{2\rho t} \int_0^t \sum_{n \in N} |g(s)Q^{1/2}e_n(x)|^2 ds &= \\ \sum_{n \in N} \int_0^t cH(2H-1)M^2e^{2\rho t} |g(s)Q^{1/2}e_n(x)|^2 ds. \end{aligned}$$

Noticing Definition 1, we have

$$\begin{aligned} E \left| \int_0^t S(t-s)g(s)dB_Q^H(s) \right|^2 &= \\ \sum_{n \in N} \int_0^t |K_H^*(S(t-\cdot)gQ^{1/2}e_n(x))(s)|^2 ds. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n \in N} \int_0^t |K_H^*(S(t-\cdot)gQ^{1/2}e_n(x))(s)|^2 ds &\leq \\ \sum_{n \in N} \int_0^t cH(2H-1)M^2e^{2\rho t} |g(s)Q^{1/2}e_n(x)|^2 ds, \end{aligned}$$

thanks to (iii), we have

$$\sum_{n \in N} \int_0^t cH(2H-1)M^2e^{2\rho t} |g(s)Q^{1/2}e_n(x)|^2 ds,$$

is uniformly convergent for $x \in [0, 1]$. Hence

$$\sum_{n \in N} \int_0^t |K_H^*(S(t-\cdot)gQ^{1/2}e_n(x))(s)|^2 ds,$$

is uniformly convergent for $x \in [0, 1]$.

On the one hand, $e_n(x)$ with respect to x is continuous, and g be an bounded linear operator, therefore

$$|K_H^*(S(t-\cdot)gQ^{1/2}e_n(x))(s)|^2,$$

with respect to x is continuous. On the other hand, thanks to the

$$\sum_{n \in N} \int_0^t |K_H^*(S(t-\cdot)gQ^{1/2}e_n(x))(s)|^2 ds$$

of uniformly convergent, so that $Z(t)$ with respect to x is continuous.

Thus the proof of Theorem 1 is completed.

3 Global existence

Set $v(t) = u(t) - Z(t)$, $t \geq 0$. Then if u satisfies

eq. (1)–(3), v satisfies

$$\frac{dv}{dt} = Av + \frac{1}{2} \frac{\partial}{\partial x} (v + Z)^2, \quad (10)$$

$$v(0) = u_0.$$

Let us write (10) as a mild form

$$v(t) = S(t)u_0 + \frac{1}{2} \int_0^t S(t-s) \frac{\partial}{\partial x} (v + Z(s))^2 ds, \quad (11)$$

which is convenient to construct the local well-posedness of the problem (10).

From now on we will study eq. (11) for a.s. $\omega \in \Omega$. We are going to solve eq. (12) by a fixed point argument in the space $C([0, T^*]; L^2(0, 1))$ on each path, i.e., we fix ω and construct a solution to a deterministic equation.

In this section, U assumed to be $L^2(0, 1)$, we have

$$g : [0, T] \rightarrow L^0_0(K, L^2(0, 1)).$$

We set

$$B_m^{T^*} = \{v \in C([0, T]; L^2(0, 1), \|v(t)\|_{L^2(0, 1)} \leq m, \forall t \in [0, T^*]\},$$

and consider the initial data that are F_0 -measurable and belong to $L^2(0, 1)$.

Then we have the following lemma.

Lemma 2 For $\|u_0\|_{L^2(0, 1)} < m$, there exists a stopping time $T^* > 0$ such that (11) has a unique solution in $B_m^{T^*}$.

Proof Take any $v \in B_m^{T^*}$ and define $y = fv$ by

$$y(t) = fv(t) = S(t)u_0 + \frac{1}{2} \int_0^t S(t-s) \frac{\partial}{\partial x} (v + Z(s))^2 ds,$$

we need to show $f : B_m^{T^*} \rightarrow B_m^{T^*}$ is a strict contraction on $B_m^{T^*}$.

$$\begin{aligned} \|fv(t)\|_{L^2(0, 1)} &\leq \|S(t)u_0\|_{L^2(0, 1)} + \\ &\frac{1}{2} \int_0^t \|S(t-s) \frac{\partial}{\partial x} (v + Z(s))^2\|_{L^2(0, 1)} ds \leq \\ &Me^{\rho t} \|u_0\|_{L^2(0, 1)} + c \int_0^t Me^{\rho t} \left\| \frac{\partial}{\partial x} (v + \right. \\ &Z(s))^2 \|_{L^2(0, 1)} ds \leq Me^{\rho T^*} \|u_0\|_{L^2(0, 1)} + \\ &cM \frac{e^{\rho T^*} - 1}{\rho} (m^2 + \mu^2) T^*, \end{aligned}$$

where $\mu = \sup_{s \in [0, t]} \|Z(s)\|_{L^2(0, 1)}$.

$$\text{Hence } \|fv(t)\|_{L^2(0, 1)} \leq m \text{ for all } t \in [0, T^*]$$

provided

$$\|u_0\|_{L^2(0, 1)} Me^{\rho T^*} + cM \frac{e^{\rho T^*} - 1}{\rho} (m^2 + \mu^2) T^* < m. \quad (12)$$

It is clear that for any $\|u_0\|_{L^2(0, 1)} < m$ there exists a T^* satisfying (12).

Now consider $v_1, v_2 \in B_m^{T^*}$ and set $y_i = fv_i$, $i = 1, 2$ and $y = y_1 - y_2$; then

$$\begin{aligned} y(t) &= \frac{1}{2} \int_0^t S(t-s) \frac{\partial}{\partial x} ((v_1 + Z(s))^2 - (v_2 + Z(s))^2) ds, \\ \|y(t)\|_{L^2(0, 1)} &\leq \frac{1}{2} \int_0^t \|S(t-s) \frac{\partial}{\partial x} ((v_1 + Z(s))^2 - \\ &(v_2 + Z(s))^2)\|_{L^2(0, 1)} ds \leq \frac{1}{2} Me^{\rho T^*} T^* \cdot \\ &\|(v_1 + Z(s))^2 - (v_2 + Z(s))^2\|_{L^2(0, 1)}. \end{aligned} \quad (13)$$

Using holder inequality, we have

$$\begin{aligned} \|(v_1 + Z(s))^2 - (v_2 + Z(s))^2\|_{L^2(0, 1)} &= \\ \|(v_1 + v_2 + 2Z(s))(v_1 - v_2)\|_{L^2(0, 1)} &\leq \\ \|(v_1 + v_2 + 2Z(s))\|_{L^2(0, 1)} \|v_1 - v_2\|_{L^2(0, 1)} &\leq \\ 2(m + \mu) \|v_1 - v_2\|_{L^2(0, 1)}. \end{aligned} \quad (14)$$

Hence

$$\|y(t)\|_{L^2(0, 1)} \leq c_1 e^{\rho T^*} T^* (m + \mu) \|v_1 - v_2\|_{L^2(0, 1)}.$$

We take T^* such that

$$c_1 e^{\rho T^*} T^* (m + \mu) < 1,$$

and (12) holds so that f is a strict contraction on $B_m^{T^*}$.

Thus we complete the proof of this lemma.

As mentioned before all the previous result are valid a.s. for $\omega \in \Omega$; in particular T^* depend on ω . In the next lemma we will show that $T^* = T$ a.s. for $\omega \in \Omega$ and hence remove the dependence on ω for the time interval on which the solution exists.

Lemma 3 If $v \in C([0, T]; L^2(0, 1))$ satisfies (10), then

$$\|v(t)\|_{L^2(0, 1)} \leq c(\mu_\infty^2 + \|u_0\|_{L^2(0, 1)}) e^{2\mu_\infty^2 t},$$

for some constant $\mu_\infty > 0$.

Proof Let $\{u_0^n\}_{n \in \mathbb{N}}$ be a sequence in $C^\infty(0, 1)$ such that $u_0^n \rightarrow u_0$ in $L^2(0, 1)$. Let $\{(B^H)^n\}_{n \in \mathbb{N}}$ be a sequence of regular processes such that

$$Z^n(t) = \int_0^t S(t-s) g(s) d(B^H)^n(s) \rightarrow Z(t),$$

in $C([0, T] \times [0, 1])$ a.s. for $\omega \in \Omega$.

Let v^n be the solution of

$$v^n(t) = S(t)u_0^n + \frac{1}{2} \int_0^t S(t-s) \frac{\partial}{\partial x} (v^n + Z^n)^2 ds,$$

given by Lemma 2. It is easy to see that v^n dose exists on an interval of time $[0, T_n]$ such that $T_n \rightarrow T^*$ and that v^n converges to v in $C([0, T^*]; L^2(0, 1))$.

Moreover v^n is regular and satisfies

$$\frac{dv^n}{dt} = Av^n + \frac{1}{2} \frac{\partial}{\partial x} (v^n + Z^n)^2, \quad v^n(0) = u_0^n, \quad (15)$$

multiplying (15) by v^n and integrating over $(0, 1)$ with respect to x , we find

$$\begin{aligned} \frac{\partial}{\partial t} \|v^n\|_{L^2(0,1)}^2 + 2 \|v^n\|_{H^1(0,1)}^2 - \\ \int_0^1 v^n \frac{\partial}{\partial x} (v^n + Z^n)^2 dx = 0. \end{aligned} \quad (16)$$

We estimate the last term in eq. (16) as follow:

$$\begin{aligned} \left| \int_0^1 v^n \frac{\partial}{\partial x} (v^n + Z^n)^2 dx \right| \leq 2 \left| \int_0^1 Z^n v^n \frac{\partial}{\partial x} v^n dx \right| + \\ \left| \int_0^1 (Z^n)^2 \frac{\partial}{\partial x} v^n dx \right| \leq 2\mu_{n,\infty}^4 + \\ 4\mu_{n,\infty}^2 \|v^n\|_{L^2(0,1)}^2 + \|v^n\|_{H^1(0,1)}^2, \end{aligned} \quad (17)$$

where $\mu_{n,\infty}^2 = \sup_{t \in [0, T]} \|Z^n(t)\|_{L^\infty(0,1)}$ for a.s. $\omega \in \Omega$.

Going back to (16) we have

$$\frac{\partial}{\partial t} \|v^n\|_{L^2(0,1)}^2 \leq 2\mu_{n,\infty}^4 + 4\mu_{n,\infty}^2 \|v^n\|_{L^2(0,1)}^2.$$

Thanks to Gronwall's lemma, we have

$$\|v^n\|_{L^2(0,1)}^2 \leq \left(\frac{\mu_{n,\infty}^2}{2} + \|u_0\|_{L^2(0,1)}^2 \right) e^{4\mu_{n,\infty}^2 t}.$$

Taking to limit as $n \rightarrow \infty$, we see that a.s.

$$\|v\|_{L^2(0,1)} \leq c(\mu_{n,\infty}^2 + \|u_0\|_{L^2(0,1)}) e^{2\mu_{n,\infty}^2 t}.$$

Thus we complete the proof of this lemma.

It is easy to derive from Lemma 1 and Lemma 2.

Theorem 2 Let u_0 be given which is F_0 -measurable and such that $u_0 \in L^2(0, 1)$ a.s.. Then there exists a unique mild solution of eq. (1)–(3), which belongs to $C([0, T]; L^2(0, 1))$ for a.s. $\omega \in \Omega$.

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基于分数布朗运动的随机微分方程

张凌浩¹, 陶祥兴^{2*}

(1. 宁波大学 理学院, 浙江 宁波 315211; 2. 浙江科技学院 理学院, 浙江 杭州 310023)

摘要: 在 Hurst 指数 $H \in (1/2, 1)$ 条件下, 研究了基于分数布朗运动的随机微分方程: $du = (Au + uu_x)dt + g(t)dB_Q^H(t)$ 的温和解的局部及整体存在性和唯一性.

关键词: 随机微分方程; 分数布朗运动; Hurst 指数

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