

Approximate Solutions of Singularly Perturbed Boussinesq Equation Using Truncated Invariant Expansion

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Abstract: Due to the complexity and nonlinearity of the nonlinear PDEs, obtaining the exact solutions is difficult. To tackle this problem, approximation methods are taken into consideration. In this paper, using the truncated invariant expansion, solving for the real nonlinear systems are proposed. Using a simple truncated invariant expansion with a universal pseudopotential for the singularly perturbed Boussinesq equation, approximate solutions of the above equation can be obtained. In some special cases, the accuracy of the solutions is found to be significantly satisfying.

Key words: approximate solution; truncated invariant expansion; Boussinesq equation

CLC number: O175.29

Document code: A

Article ID: 1001-5132 (2012) 04-0127-06

There are some marvelous methods to study the integrability of a nonlinear partial differential equation (PDE). The Painlevé analysis developed by Weiss et al^[1] is one of the most effective approaches. Applying the WTC approach to nonlinear PDEs, one can obtain not only properties like the Painlevé property, Lax pair, bilinear form, Bäcklund transformation of integrable models but also solutions both for integrable and non-integrable models^[2-6].

In [7], Conte had proposed a simplification of the WTC approach which is corresponding to the re-summation of the usual WTC approach such that the new expansion coefficients are all invariant under the Möbius transformation. The truncated expansion approach in Conte's analysis is related to a special type of non-truncated summation in the usual WTC approach^[8]. In [9], Pickering proposed a nonstandard truncation expansion approach basing on Conte's Painlevé

expansion. Then some new nontrivial solutions may be obtained due to Pickering's nonstandard truncation approach. In Pickering's consideration, the truncation expansion will yield a new nontrivial solution only for those equations which possess more than one branch in the original WTC analysis. Similar to Conte's consideration, we may obtain some other types of expansions to study the Painlevé property. Perhaps, the different uses of the expansions may cause complexity in the study of the Painlevé analysis. However, it is useful to get different new solutions.

In next section, we simply review the general aspect of the Conte's Painlevé expansion. In section 3, due to the complexity and nonlinearity of some nonlinearity PDEs, it is difficult to give some exact solutions. So we use the truncated invariant expansion to solve real PDEs and find out some reasonable approximations of real physics^[10-12]. Taking the

Received date: 2012-03-09.

JOURNAL OF NINGBO UNIVERSITY (NSEE): <http://3xb.nbu.edu.cn>

Foundation items: Supported by the National Natural Science Foundation of China (11175092); Student Research Innovation Program (SRIP) of Ningbo University.

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singularly perturbed Boussinesq equation as a simple example to show how the expansion yields some more reasonable approximate solutions. In some special cases, if the arbitrary constants are selected suitably, the approximate solutions may become exact. And some representative profiles of the solutions are graphically displayed. The last section is a short summary.

1 Invariant Painlevé expansions by Conte

For a given PDE, says

$$F(u, u_x, u_t, \dots, u_{x^n}, u_{t^n}) = 0, \quad (1)$$

the invariant Painlevé expansion takes the form

$$u = \sum_{i=0}^{\infty} u_i \xi^{i-\alpha}. \quad (2)$$

With the pseudopotential ξ of the given form

$$\xi_x = 1 + \frac{S\xi^2}{2}, \quad (3)$$

$$\xi_t = -C + C_x \xi - \frac{(SC + C_{xx})\xi^2}{2}. \quad (4)$$

The compatibility condition of eq. (3) and eq. (4) has the following form

$$S_t + 2C_x S + C_{xxx} + CS_x = 0, \quad (5)$$

eq. (5) possesses a special solution

$$S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2} \frac{\phi_{xx}^2}{\phi_x^2}, \quad C = -\frac{\phi_t}{\phi_x}. \quad (6)$$

And S , C satisfies with Mobius transformation invariance,

$$\phi \rightarrow \frac{a+b\phi}{c+d\phi}, \quad ab \neq cd.$$

In next section, we concentrate on taking the singularly perturbed Boussinesq equation as an example, the truncated invariant expansion is used to find out some approximate solutions. In some special cases, the approximate solutions are just the exact solutions of singularly perturbed Boussinesq equation.

2 The truncated invariant expansion

The singularly perturbed Boussinesq equation^[13]

$$u_{tt} + u_{xx} + 2u_x^2 + 2uu_{xx} + u_{xxxx} + \nu u_{xxxxx} = 0, \quad (7)$$

where ν is arbitrary constant.

Using the leading order analysis first, substituting eq. (7) into eq. (2), we can find that eq. (2) should have the form

$$u = \frac{u_0}{\xi^4} + \frac{u_1}{\xi^3} + \frac{u_2}{\xi^2} + \frac{u_3}{\xi} + u_4. \quad (8)$$

In order to balance the effects of the nonlinearity, and those of the dispersion in positive and negative directions. Substituting the truncated expansion eq. (8) into the singularly perturbed Boussinesq equation and canceling the coefficients of ξ_i ($i = -9, -8, -7, -6, -5$), yields five further constraints

$$u_0(u_0 + 840\nu) = 0, \quad (9)$$

$$112u_0u_1 - 32u_0u_{0x} - 40320\nu u_{0x} + 20160\nu u_1, \quad (10)$$

$$42u_1^2 + 97440\nu u_0S - 28u_{0x}u_1 + 2u_{0x}^2 + 12600\nu u_{0xx} + 84u_0u_2 + 64u_0^2S - 28u_0u_{1x} - 15120\nu u_{1x} + 840u_0 + 5040\nu u_2 + 2u_0u_{0xx} = 0, \quad (11)$$

$$2u_0u_{1xx} + 4u_{0x}u_{1x} - 24u_{0x}u_2 - 4320\nu u_{2x} - 4u_0^2S_x - 57600\nu u_{0x}S + 360u_1 + 2u_1u_{0xx} - 22200\nu u_0S_x + 60u_1u_2 + 720\nu u_3 - 2400\nu u_{0xxx} - 480u_{0x} + 29880\nu u_1S - 16u_{0x}u_0S - 24u_0u_{2x} - 24u_1u_{1x} + 5400\nu u_{1xx} + 60u_0u_3 + 98u_0u_1S = 0, \quad (12)$$

$$20u_0 + 120u_2 - 240u_{1x} + 20u_2^2 + 2u_{1x}^2 + 1040u_0S + 4u_{0x}u_{2x} - 20u_{0x}u_3 - 20u_0u_{3x} - 20u_{1x}u_2 - 20u_1u_{2x} + 40u_1u_3 + 40u_4u_0 + 36u_1^2S + 14u_0^2S^2 + 20u_0C^2 - 720\nu u_{3x} - 14u_{0x}u_1S - 14u_0u_{1x}S + 72u_0u_2S + 57680\nu u_0S^2 - 19800\nu u_{1x}S - 7200\nu u_1S_x + 6720\nu u_2S + 12600\nu u_{0x}S_x - 7u_0u_1S_x + 3720\nu u_0S_{xx} + 15600\nu u_{0xx}S + 2u_0u_{2xx} + 2u_1u_{1xx} + 2u_2u_{0xx} - 1200\nu u_{1xxx} + 300\nu u_{0xxx} + 1800\nu u_{2xx} + 120u_{0xx} = 0. \quad (13)$$

According to the above five further constraints, it is easy to find out all possible solutions of the over-determined constraint equations.

As in truncated expansion approach, we consider only the $u_0 \neq 0$ on these constraint equations. After a series of complicated calculations and some simplifications, the final result can be written as

$$u_0 = -840\nu, \quad u_1 = 0, \quad (14)$$

$$u_2 = -560\nu S - 140/13, \quad (15)$$

$$u_3 = 140\nu S_x, \quad (16)$$

$$u_4 = -\frac{1}{1014\nu}(507\nu C^2 + 28392\nu^2 S_{xx} + 507\nu + 3640\nu S + 66248\nu^2 S^2 - 31). \quad (17)$$

2.1 The first kind approximate solution

When $u_4 = 0$, substituting eq. (6) into eq. (17), we can get

$$\begin{aligned} & 507\nu\phi_{xxxx} - 31\phi_{xxxx} + 3640\nu\phi_{xxx}\phi_x^3 - \\ & 5460\nu\phi_x^2\phi_{xx}^2 - 47320\nu^2\phi_{xxx}^2\phi_x^2 + \\ & 283920\nu^2\phi_{xxx}\phi_{xx}^2\phi_x - 106470\nu^2\phi_{xx}^4 + \\ & 28392\nu^2\phi_{xxxx}\phi_x^3 - 141960\nu^2\phi_{xxx}\phi_{xx}\phi_x^2 + \\ & 507\nu\phi_x^2\phi_x^2 = 0. \end{aligned} \quad (18)$$

Obviously, $\phi = e^{(kx+ot)}$ is the solution of the above eq. (18), where k is arbitrary constant,

$$\omega = \pm \frac{\sqrt{-3\nu(16562\nu^2 k^4 - 1820\nu k^2 - 31 + 507\nu)k}}{39\nu}. \quad (19)$$

Let $\phi = e^{(kx+ot)}$ do Mobious transformation, taking the form

$$\phi_1 = \frac{a + be^{\frac{[kx \pm \frac{\sqrt{-3\nu(16562\nu^2 k^4 - 1820\nu k^2 - 31 + 507\nu)k}}{39\nu}]}}}{c + de^{\frac{[kx \pm \frac{\sqrt{-3\nu(16562\nu^2 k^4 - 1820\nu k^2 - 31 + 507\nu)k}}{39\nu}]}}}. \quad (20)$$

Since ϕ satisfies Mobious transformation invariance, we can see that ϕ_1 is also the solution of the above equation eq. (18).

Until now, u_0, u_1, u_2, u_3, u_4 are all known to us, next we only need find out ξ . We can get our wanted answer from eq. (3), eq. (4) and eq. (6),

$$\xi = \frac{2}{k} \coth\left[\frac{kx}{2} \pm \frac{\sqrt{-3\nu(16562\nu^2 k^4 - 1820\nu k^2 - 31 + 507\nu)k}}{78\nu} + t_0\right]. \quad (21)$$

Substituting $u_0, u_1, u_2, u_3, u_4, \xi$, eqs. (6), (20) into eq. (8), the approximate solution can be written as

$$\begin{aligned} u = & -\frac{35}{26} \tanh[(\pm 39kx + \\ & \sqrt{-49686k^4\nu^3 + 5460\nu^2 k^2 + 93\nu - 1521\nu^2} kt \pm \\ & 78t_0\nu)/78\nu]^2 k^2 \{2 - 52\nu k^2 + 39\nu \tanh[(\pm 39kx + \\ & \sqrt{-49686k^4\nu^3 + 5460\nu^2 k^2 + 93\nu - 1521\nu^2} kt \pm \end{aligned}$$

$$78t_0\nu)/78\nu]^2 k^2\}, \quad (22)$$

where k, ν, t_0 are all arbitrary constants.

Substituting eq. (22) into eq. (7), we know that

$$\begin{aligned} & \{2 \cosh[(\pm 39kx + \\ & \sqrt{-3\nu(16562k^4\nu^2 - 1820\nu k^2 - 31 + 507\nu)k} t \pm \\ & 78t_0\nu)/78\nu]^2 - 3\} / \cosh[(\pm 39kx + \\ & \sqrt{-3\nu(16562k^4\nu^2 - 1820\nu k^2 - 31 + 507\nu)k} t \pm \\ & 78t_0\nu)/78\nu]^4 35k^4 (13\nu k^2 + 1)(1690k^4\nu^2 - \\ & 403\nu k^2 + 31) = 0. \end{aligned} \quad (23)$$

It is easy to find out the relationship between ν and k ,

$$k = \pm \frac{1}{\sqrt{-13\nu}}. \quad (24)$$

Substituting eq. (24) into eq. (22), the result is the exact solution of the singularly perturbed Boussinesq equation, reads

$$\begin{aligned} u = & -\frac{105}{338} \tanh((\pm 13x\nu \pm \sqrt{-169\nu^2 - 69\nu} t + \\ & 26t_0\nu\sqrt{-13\nu}) / (26\nu\sqrt{-13\nu}))^2 [\tanh((\pm 13x\nu \pm \\ & \sqrt{-169\nu^2 - 69\nu} t + 26t_0\nu\sqrt{-13\nu}) / \\ & (26\nu\sqrt{-13\nu}))^2 - 2], \end{aligned} \quad (25)$$

where k, ν, t_0 are all arbitrary constants.

And let $\nu = -1/13$, $t_0 = 0$, thus we can get two exact solutions

$$u = -\frac{105}{26} \frac{\cosh(\frac{x}{2} + \frac{\sqrt{182}t}{13})^4 - 1}{\cosh(\frac{x}{2} + \frac{\sqrt{182}t}{13})^4}, \quad (26)$$

or

$$u = -\frac{105}{26} \frac{\cosh(-\frac{x}{2} + \frac{\sqrt{182}t}{13})^4 - 1}{\cosh(-\frac{x}{2} + \frac{\sqrt{182}t}{13})^4}. \quad (27)$$

When $k \neq 1$, since k is arbitrary, in this case, we have an infinite number of approximate solutions. But we only need find a more reasonable approximation of the solution. After analysis, the closer to 1, the selected k is, the more precisely approximate solution we will get. Taking into account the convenience of calculating at the same time, let $k = 6/7$, we get two approximate solutions

$$u = -\frac{180}{4459} [\cosh(\frac{3x}{7} + \frac{\sqrt{163878}t}{637})^2 - 1] \cdot [67 \cosh(-\frac{3x}{7} + \frac{\sqrt{163878}t}{637})^2 + 54] / \cosh(\frac{3x}{7} + \frac{\sqrt{163878}t}{637})^4, \quad (28)$$

or

$$u = -\frac{180}{4459} [\cosh(-\frac{3x}{7} + \frac{\sqrt{163878}t}{637})^2 - 1] \cdot [67 \cosh(-\frac{3x}{7} + \frac{\sqrt{163878}t}{637})^2 + 54] / \cosh(-\frac{3x}{7} + \frac{\sqrt{163878}t}{637})^4. \quad (29)$$

2.2 The second kind approximate solution

When $u_3 = 0$, $u_4 = 0$, substituting eq. (6) into eq. (16), it is easy to get for us

$$\phi_{xxxx} - 4\phi_{xxx}\phi_x + 3\phi_{xx}^2 = 0. \quad (30)$$

In the same way, $\phi = e^{(kx+\omega t)}$ is also the solution of the above equation, so we can get ω , but the got ω is arbitrary, let ϕ do Mobious transformation

$$\phi_2 = \frac{a + be^{(kx+\omega t)}}{c + de^{(kx+\omega t)}}, \quad (31)$$

with the same point of view, ϕ_2 satisfies eq. (30).

Next, we only need find out ξ substituting eq. (6) and ϕ_2 into eq. (3), eq. (4), we know that

$$\xi = \frac{2}{k} \coth(\frac{kx}{2} + \frac{\omega t}{2} - t_0). \quad (32)$$

Now substituting $u_0, u_1, u_2, u_3, u_4, \xi$, eq. (6), eq. (31) into eq. (8), says

$$u = -\frac{35}{26} [39\nu k^2 \tanh(\frac{kx}{2} + \frac{\omega t}{2} - t_0)^2 + 2 - 52\nu k^2] \tanh(\frac{kx}{2} + \frac{\omega t}{2} - t_0)^2 k^2, \quad (33)$$

where k, ν, t_0 are all arbitrary constants. Since ω is arbitrary, let $k = 6/7$, $\nu = -1/13$, $\omega = 6/7$, $t_0 = 0$, we have a reasonable approximate solution

$$u = -\frac{180}{4459} \frac{[\cosh(3x/7 + 3t/7)^2 - 1]}{\cosh(3x/7 + 3t/7)^4} \cdot \frac{[67 \cosh(3x/7 + 3t/7)^2 + 54]}{\cosh(3x/7 + 3t/7)^4}. \quad (34)$$

2.3 The third kind approximate solution

When $u_3 = 0$, $u_4 \neq 0$, in accordance with the same

principle of the case when $u_3 = 0$, $u_4 = 0$, substituting u_0, u_1, u_2, u_3, u_4 , eq. (6), eq. (31) and eq. (32) into eq. (8), the final approximate solution can be written as

$$u = -\frac{1}{1014k^2\nu} [507\nu k^2 + 2730 \tanh(\frac{kx}{2} + \frac{\omega t}{2} - t_0)^2 k^4 \nu - 1820\nu k^4 + 507\nu \omega^2 + 53235\nu^2 \tanh(\frac{kx}{2} + \frac{\omega t}{2} - t_0)^4 k^6 + 16562\nu^2 k^6 - 70980 \tanh(kx/2 + \omega t/2 - t_0)^2 k^6 \nu^2 - 31k^2], \quad (35)$$

where k, ν, t_0 are all arbitrary constants. Since ω is arbitrary, after some simplifications, we can get an approximate solution, when

$$k = 6/7, \nu = -1/13, \omega = 6/7, t_0 = 0,$$

that is to say

$$u = -\frac{56323 \cosh(3x/7 + 3t/7)^4}{26754 \cosh(3x/7 + 3t/7)^4} - \frac{14040 \cosh(3x/7 + 3t/7)^2 - 58320}{26754 \cosh(3x/7 + 3t/7)^4}. \quad (36)$$

Substituting eq. (35) into eq. (7), we know

$$-\frac{35}{13182} k^4 [\tanh(\frac{kx}{2} + \frac{\omega t}{2} - t_0) - 1] (13\nu k^2 + 1) \cdot (1690\nu^2 k^4 - 403\nu k^2 + 31) [3 \tanh(\frac{kx}{2} + \frac{\omega t}{2} - t_0)^2 - 1] = 0. \quad (37)$$

From the above equation, k can be expressed with ν ,

$$k = \pm 1 / \sqrt{-13\nu}. \quad (38)$$

Substituting eq. (38) into eq. (35), exact solution can be written as

$$u = -\frac{1}{338\nu} [169\nu - 210 \tanh((x + \omega t \sqrt{-13\nu} + 2t_0 \sqrt{-13\nu}) / (2\sqrt{-13\nu}))^2 + 69 - 2197\nu^2 \omega^2 + 105 \tanh(\frac{x + \omega t \sqrt{-13\nu} + 2t_0 \sqrt{-13\nu}}{2\sqrt{-13\nu}})^4], \quad (39)$$

Let $\nu = -1/13$, $\omega = 6/7$, $t_0 = 0$, we can get an exact solution

$$u = \frac{1138}{637} - \frac{105}{13} \tanh(\frac{x}{2} + \frac{3t}{7})^2 + \frac{105}{26} \tanh(\frac{x}{2} + \frac{3t}{7})^4. \quad (40)$$

In approximate solutions (28), (29), (34), (36) and

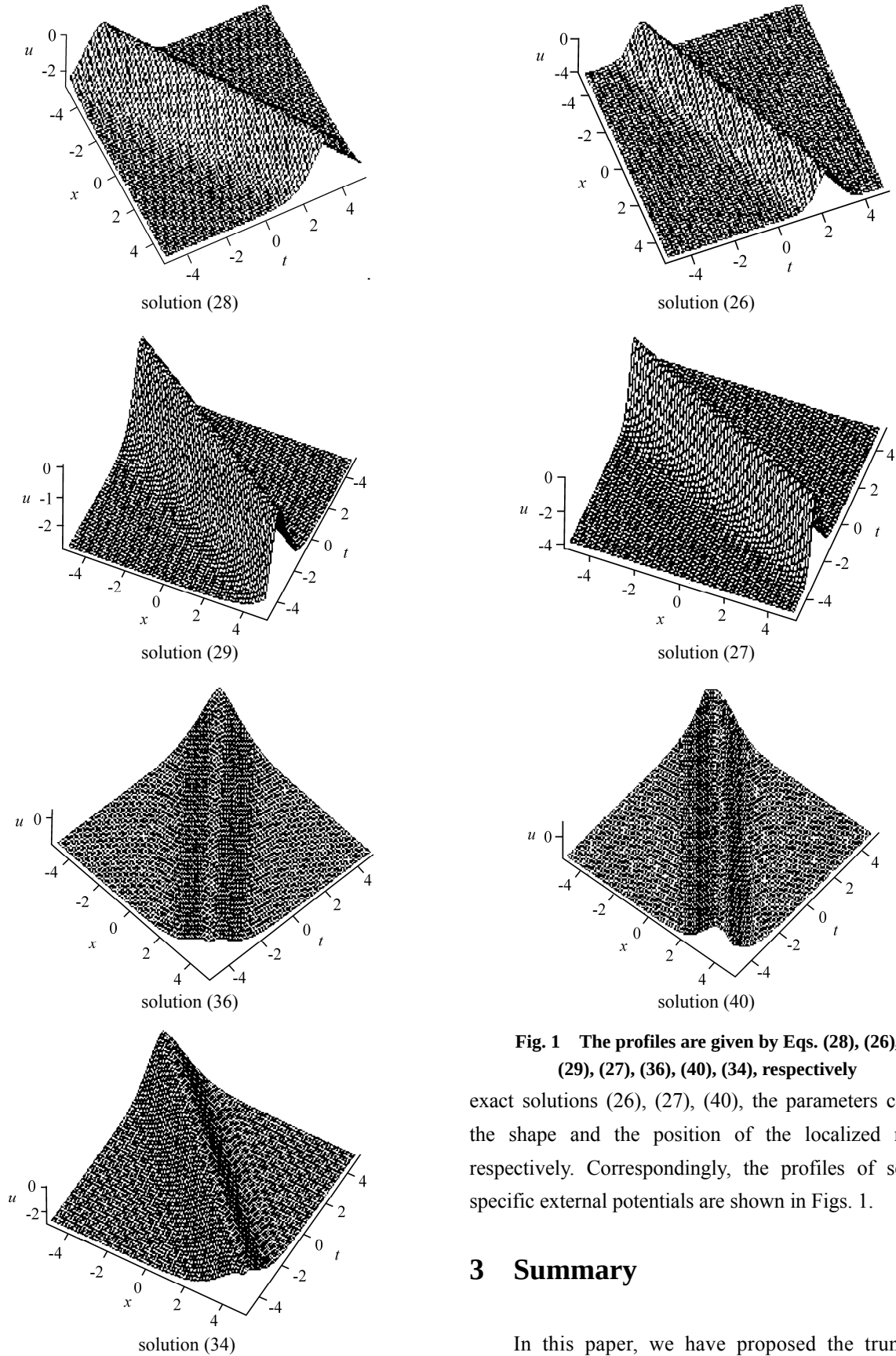


Fig. 1 The profiles are given by Eqs. (28), (26), (29), (27), (36), (40), (34), respectively

exact solutions (26), (27), (40), the parameters control the shape and the position of the localized mode, respectively. Correspondingly, the profiles of several specific external potentials are shown in Figs. 1.

3 Summary

In this paper, we have proposed the truncated

invariant expansion to solve integrable or non-integrable nonlinear models. Taking the singularly perturbed Boussinesq equation as a concrete example, we have obtained three kinds approximate solution of this model. In some special cases, if the arbitrary constants are selected suitably, the approximate solutions may become exact. Moreover, applying the same idea to other common nonlinear equations, we can also find this quite universal expansion approach which is valid for various nonlinear PDEs.

Acknowledgements

The author is indebted for the helpful discussions with Professor LOU Sen-yue and Doctor JIA Man.

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通过截断不变展开求奇异扰动 Boussinesq 方程的近似解

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摘要: 由于非线性偏微分方程的复杂性和非线性性, 很难求出它们的准确解, 因此某些合理近似以及通过截断不变展开求解实非线性系统是被考虑和建议的. 文中一个简单截断不变展开及其一个普遍的势被用于奇异扰动 Boussinesq 方程, 可以得到该方程的近似解, 在某些情况下, 这些解亦为准确解.

关键词: 近似解; 截断不变展开; Boussinesq 方程

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