

非线性 Schrödinger 系统整体解的 Liouville 定理

吴千秋, 计 婷, 胡良根*

(宁波大学 理学院, 浙江 宁波 315211)

摘要: 本文考虑具有 Hardy 位势的非线性 Schrödinger 系统, 分别利用变量替换、测试函数和积分估计证明了系统非负经典整体解的 Liouville 型定理成立.

关键词: Schrödinger 系统; Liouville 型定理; 替换; 测试函数

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1 引言

考虑具有 Hardy 位势的非线性 Schrödinger 系统:

$$\begin{cases} -\Delta u = \frac{l}{|x|^2} u - |x|^\alpha |v|^{p-1} v, \\ -\Delta v = \frac{n}{|x|^2} v - |x|^\beta |u|^{q-1} u, \end{cases} \quad x \in \mathbb{R}^N, \quad (1)$$

其中, $\alpha, \beta \geq 0, p, q > 1$ 和 $l, n \in (-\infty, (N-2)^2/4)$. 我们建立 Liouville 型定理, 即系统(1)式不存在非负经典解.

近年来, 在原点具有奇性的非线性 Schrödinger 方程引起了很多专家关注^[1-5]. 例如, Jeong 等^[4]和 Du 等^[5]分别讨论了具有 Hardy 位势的非线性 Schrödinger 方程稳定解和有限 Morse 指标解的 Liouville 型定理.

研究系统(1)式的目的是源自于著名文献[6], 在此文中, Gidas 和 Spruck 建立 Lane-Emden 方程的 Liouville 型定理, 且 Liouville 型定理在研究椭圆方程和抛物方程时有广泛的应用. 特别, 联合 Gidas 和 Spruck 的 rescaling 方法或爆破方法^[7], 无变分结构的 Dirichlet 问题正解的先验估计和存在性被建立. 2010 年, Dancer 等^[8]利用 Liouville 定理、爆破分析和变分法研究了一类非线性 Schrödinger 方程孤子波解的先验估计和存在性. 另一方面, 许多专家把 Lane-Emden 方程的 Liouville 型定理作为一个

猜想进行了广泛的研究^[9-11].

但对具有 Hardy 位势的非线性 Schrödinger 系统而言, 因为位势项的出现会致使研究其解的 Liouville 型性质更加困难, 此类问题的研究与可积系统也有很大的关系^[12]. 采用 Souplet 的方法, Phan 等^[13]和 Fazly 等^[14]在 $N=3$ 情况下, 分别建立了 Hardy-Hénon 方程和 Hénon-Lane-Emden 系统解的 Liouville 型定理. 也需要注意到, Yarur^[15]利用球面平均方法研究了一类加权椭圆系统正解的非存在性.

受文献[5,16-17]的启发, 本文目的是利用变量替换、测试函数和积分估计方法证明具有 Hardy 位势非线性 Schrödinger 系统解的 Liouville 型定理.

定理 1 设 (u, v) 是系统(1)式的非负经典解, $\alpha, \beta \geq 0, p, q > 1$ 和 $l, n \in (-\infty, (N-2)^2/4)$. 如果下面 2 个条件之一成立:

$$\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 - l} - \frac{\alpha + p\beta + 2p + 2}{pq - 1} < 0, \quad (2)$$

或

$$\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 - n} - \frac{\beta + q\alpha + 2q + 2}{pq - 1} < 0, \quad (3)$$

则 $u = v \equiv 0$.

注 1 本文并没有要求系统(1)式的解是径向对称的.

定义 如果系统解是在 $\mathbb{R}^N$ 上, 则称为整体解.

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第一作者: 吴千秋 (1993-), 女, 河南信阳人, 在读硕士研究生, 主要研究方向: 非线性泛函分析. E-mail: wuqianqiu123@163.com

*通信作者: 胡良根 (1977-), 男, 江西临川人, 副教授, 主要研究方向: 非线性泛函分析. E-mail: hulianggen@tom.com

记号 全文使用 $B_R(x)$ 记为以 x 为中心, R 为半径的开球, 也写 $B_R = B_R(0)$. C 记为与 u, v, ω 和 z 无关的一般正常数, 这里 $u = u(x), v = v(x), \omega = \omega(x), z = z(x)$.

2 定理 1 的证明

记

$$\theta := -2\sigma, \quad \sigma := \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - l},$$

$$\tau := -2\delta, \quad \delta := \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - n},$$

定义变量替换 $\omega(x) = |x|^\sigma u(x)$ 及 $z(x) = |x|^\delta v(x)$, 则代入系统(1)式可得:

$$\Delta u + \frac{l}{|x|^2} u = \operatorname{div}(|x|^\theta D\omega) |x|^\sigma,$$

$$\Delta v + \frac{n}{|x|^2} v = \operatorname{div}(|x|^\tau Dz) |x|^\delta.$$

联合系统(1)式意味着:

$$\begin{cases} \operatorname{div}(|x|^\theta D\omega) = |x|^\lambda |z|^{p-1} z, \\ \operatorname{div}(|x|^\tau Dz) = |x|^\mu |\omega|^{q-1} \omega, \end{cases} \quad (4)$$

其中, $\lambda = \alpha - \sigma - \delta p$, $\mu = \beta - \delta - \sigma q$. 则可发现上面变化过程为可逆, 因此系统(1)式和加权椭圆系统(4)式是等价的.

全文选择一个具有 $0 < k < \min\{p-1, q-1\}$ 关系的常数 k 和具有 $0 \leq \phi \leq 1$ 关系的测试函数 $\phi \in C_0^1(R^N)$,

$$\phi(x) = \begin{cases} 1, & \text{如果 } |x| < R, \\ 0, & \text{如果 } |x| > 2R, \end{cases} \quad \text{和 } |D\phi| < \frac{C}{R}.$$

用 $\omega^k \phi$ 乘以系统(4)式的第 1 个方程, 并分部积分可得:

$$\int_{R^N} |x|^\lambda z^p \omega^k \phi dx = -k \int_{R^N} |x|^\theta |D\omega|^2 \omega^{k-1} \phi dx - \int_{R^N} |x|^\theta D\omega \cdot D\phi \omega^k dx. \quad (5)$$

对(5)式的第 2 项应用 Cauchy 不等式, 可得:

$$\int_{R^N} |x|^\theta D\omega \cdot D\phi \omega^k dx \leq \frac{k}{2} \int_{R^N} |x|^\theta |D\omega|^2 \omega^{k-1} \phi dx + \frac{1}{2k} \int_{R^N} |x|^\theta |D\phi|^2 \omega^{k+1} \phi^{-1} dx,$$

且联合(5)式产生

$$\int_{R^N} |x|^\lambda z^p \omega^k \phi dx + \frac{k}{2} \int_{R^N} |x|^\theta |D\omega|^2 \omega^{k-1} \phi dx \leq$$

$$C \int_{R^N} |x|^\theta |D\phi|^2 \omega^{k+1} \phi^{-1} dx. \quad (6)$$

类似于上面的讨论可得:

$$\int_{R^N} |x|^\mu \omega^q z^k \phi dx + \frac{k}{2} \int_{R^N} |x|^\tau |Dz|^2 z^{k-1} \phi dx \leq C \int_{R^N} |x|^\tau |D\phi|^2 z^{k+1} \phi^{-1} dx. \quad (7)$$

用 ϕ 乘以系统(4)式, 分部积分和应用 Hölder 不等式可得:

$$\int_{R^N} |x|^\lambda z^p \phi dx = - \int_{R^N} |x|^\theta D\omega D\phi dx \leq \left(\int_{R^N} |x|^\theta |D\omega|^2 \omega^{k-1} \phi dx \right)^{\frac{1}{2}} \cdot \left(\int_{R^N} |x|^\theta |D\phi|^2 \omega^{1-k} \phi^{-1} dx \right)^{\frac{1}{2}},$$

$$\int_{R^N} |x|^\mu \omega^q \phi dx \leq \left(\int_{R^N} |x|^\theta |D\omega|^2 \omega^{k-1} \phi dx \right)^{\frac{1}{2}} \cdot \left(\int_{R^N} |x|^\tau |D\phi|^2 \omega^{1-k} \phi^{-1} dx \right)^{\frac{1}{2}}.$$

因此, 从(6)式和(7)式意味着

$$\int_{R^N} |x|^\lambda z^p \phi dx \leq C \left(\int_{R^N} |x|^\theta |D\phi|^2 \omega^{k+1} \phi^{-1} dx \right)^{\frac{1}{2}} \cdot \left(\int_{R^N} |x|^\theta |D\phi|^2 \omega^{1-k} \phi^{-1} dx \right)^{\frac{1}{2}}, \quad (8)$$

$$\int_{R^N} |x|^\mu \omega^q \phi dx \leq C \left(\int_{R^N} |x|^\tau |D\phi|^2 z^{k+1} \phi^{-1} dx \right)^{\frac{1}{2}} \cdot \left(\int_{R^N} |x|^\tau |D\phi|^2 z^{1-k} \phi^{-1} dx \right)^{\frac{1}{2}}. \quad (9)$$

对(8)式右边第 1 项使用 Hölder 不等式导致

$$\int_{R^N} |x|^\theta |D\phi|^2 \omega^{k+1} \phi^{-1} dx \leq \left(\int_{R^N} |x|^\mu \omega^q \phi dx \right)^{\frac{1+k}{q}} \cdot \left(\int_{R^N} |x|^{a_1} |D\phi|^{b_1} \phi^{c_1} dx \right)^{\frac{q-k-1}{q}},$$

其中,

$$a_1 = \left(\theta - \frac{\mu(k+1)}{q} \right) \frac{q}{q-k-1}, b_1 = \frac{2q}{q-k-1},$$

$$c_1 = -\frac{q+k+1}{q-k-1}.$$

类似可得:

$$\int_{R^N} |x|^\theta |D\phi|^2 \omega^{1-k} \phi^{-1} dx \leq \left(\int_{R^N} |x|^\mu \omega^q \phi dx \right)^{\frac{1-k}{q}} \cdot \left(\int_{R^N} |x|^{a_2} |D\phi|^{b_2} \phi^{c_2} dx \right)^{\frac{q+k-1}{q}},$$

其中,

$$a_2 = \left(\theta - \frac{\mu(1-k)}{q} \right) \frac{q}{q+k-1}, b_2 = \frac{2q}{q+k-1},$$

$$c_2 = -\frac{q+1-k}{q+k-1}.$$

现在将以上 2 个不等式代入(8)式, 则

$$\int_{R^N} |x|^\lambda z^p \phi dx \leq C \left(\int_{R^N} |x|^\mu \omega^q \phi dx \right)^{\frac{1}{q}}.$$

$$M_1^{\frac{q-k-1}{2q}} M_2^{\frac{q+k-1}{2q}}, \quad (10)$$

其中,

$$M_1 = \int_{R^N} |x|^{a_1} |D\phi|^{b_1} \phi^{c_1} dx,$$

$$M_2 = \int_{R^N} |x|^{a_2} |D\phi|^{b_2} \phi^{c_2} dx.$$

采用以上同样的计算方式, 可发现

$$\int_{R^N} |x|^\tau |D\phi|^2 z^{k+1} \phi^{-1} dx \leq \left(\int_{R^N} |x|^\lambda z^p \phi dx \right)^{\frac{k+1}{p}}.$$

$$\left(\int_{R^N} |x|^{r_1} |D\phi|^{s_1} \phi^{t_1} dx \right)^{\frac{p-k-1}{p}},$$

$$\int_{R^N} |x|^\tau |D\phi|^2 z^{1-k} \phi^{-1} dx \leq \left(\int_{R^N} |x|^\lambda z^p \phi dx \right)^{\frac{1-k}{p}}.$$

$$\left(\int_{R^N} |x|^{r_2} |D\phi|^{s_2} \phi^{t_2} dx \right)^{\frac{p+k-1}{p}}.$$

其中,

$$r_1 = \left(\tau - \frac{\lambda(k+1)}{p} \right) \frac{p}{p-k-1}, s_1 = \frac{2p}{p-k-1},$$

$$t_1 = -\frac{p+k+1}{p-k-1},$$

$$r_2 = \left(\tau - \frac{\lambda(1-k)}{p} \right) \frac{p}{p+k-1}, s_2 = \frac{2p}{p+k-1},$$

$$t_2 = -\frac{p+1-k}{p+k-1}.$$

则把以上 2 个不等式代入(9)式, 可得:

$$\int_{R^N} |x|^\mu \omega^q \phi dx \leq C \left(\int_{R^N} |x|^\lambda z^p \phi dx \right)^{\frac{1}{p}}.$$

$$M_3^{\frac{p-k-1}{2p}} M_4^{\frac{p+k-1}{2p}}, \quad (11)$$

其中,

$$M_3 = \int_{R^N} |x|^{r_1} |D\phi|^{s_1} \phi^{t_1} dx,$$

$$M_4 = \int_{R^N} |x|^{r_2} |D\phi|^{s_2} \phi^{t_2} dx.$$

联合(10)式和(11)式可发现

$$\int_{R^N} |x|^\lambda z^p \phi dx \leq C \left(\int_{R^N} |x|^\lambda z^p \phi dx \right)^{\frac{1}{pq}} M_1^{\frac{q-k-1}{2q}}.$$

$$M_2^{\frac{q+k-1}{2q}} M_3^{\frac{p-k-1}{2pq}} M_4^{\frac{p+k-1}{2pq}} \leq$$

$$CM_1^{\frac{p(q-k-1)}{2(pq-1)}} M_2^{\frac{p(q+k-1)}{2(pq-1)}} M_3^{\frac{p-k-1}{2(pq-1)}} M_4^{\frac{p+k-1}{2(pq-1)}}. \quad (12)$$

$$\int_{R^N} |x|^\mu \omega^q \phi dx \leq CM_1^{\frac{q-k-1}{2(pq-1)}} M_2^{\frac{q+k-1}{2(pq-1)}}.$$

$$M_3^{\frac{q(p-k-1)}{2(pq-1)}} M_4^{\frac{q(p+k-1)}{2(pq-1)}}. \quad (13)$$

因此从(12)式和(13)式可得:

$$\int_{R^N} |x|^\lambda z^p \phi dx \leq C(R^{N+a_1-b_1})^{\frac{p(q-k-1)}{2(pq-1)}} (R^{N+a_2-b_2})^{\frac{p(q+k-1)}{2(pq-1)}}.$$

$$(R^{N+r_1-s_1})^{\frac{p-k-1}{2(pq-1)}} (R^{N+r_2-s_2})^{\frac{p+k-1}{2(pq-1)}},$$

$$\int_{R^N} |x|^\mu \omega^q \phi dx \leq C(R^{N+a_1-b_1})^{\frac{q(p-k-1)}{2(pq-1)}} (R^{N+a_2-b_2})^{\frac{q(p+k-1)}{2(pq-1)}}.$$

$$(R^{N+r_1-s_1})^{\frac{q(p-k-1)}{2(pq-1)}} (R^{N+r_2-s_2})^{\frac{q(p+k-1)}{2(pq-1)}}.$$

直接计算上面不等式的指数可得:

$$N \left[\frac{p(q-k-1)}{2(pq-1)} + \frac{p(q+k-1)}{2(pq-1)} + \frac{p-k-1}{2(pq-1)} + \frac{p+k-1}{2(pq-1)} \right] +$$

$$(a_1-b_1) \frac{p(q-k-1)}{2(pq-1)} + (a_2-b_2) \frac{p(q+k-1)}{2(pq-1)} +$$

$$(r_1-s_1) \frac{p-k-1}{2(pq-1)} + (r_2-s_2) \frac{p+k-1}{2(pq-1)} =$$

$$\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2} \right)^2} - l - \frac{\alpha + p\beta + 2p + 2}{pq-1}.$$

这意味着

$$\int_{R^N} |x|^\lambda z^p \phi dx \leq CR^{\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2} \right)^2} - l - \frac{\alpha + p\beta + 2p + 2}{pq-1}}.$$

类似于以上计算, 可得:

$$\int_{R^N} |x|^\mu \omega^q \phi dx \leq CR^{\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2} \right)^2} - n - \frac{\beta + q\alpha + 2q + 2}{pq-1}}.$$

如果条件(2)成立, 则当 $R \rightarrow +\infty$ 时, 可得:

$$\int_{B_R} |x|^\lambda z^p dx \rightarrow 0,$$

即 $z \equiv 0$. 由(4)式表明 $\omega \equiv 0$, 因此由解的连续性和系统(1)与系统(4)的等价性容易推出 $u = v \equiv 0$. 类似地, 如果条件(3)式成立, 则可以得到同样的结果.

注 2 定理 1 的证明方法同样适用于下面系统

$$\begin{cases} -\Delta u = \frac{l}{|x|^2} u - f(x) |v|^{p-1} v, \\ -\Delta v = \frac{n}{|x|^2} v - g(x) |u|^{q-1} u, \end{cases} \quad x \in R^N.$$

其中, 当 $|x| \rightarrow \infty$ 时, $f(x) \sim |x|^\alpha$ 和 $g(x) \sim |x|^\beta$, 而 $\alpha, \beta, p, q, l, n$ 与定理 1 相同. 如果条件(2)式和(3)式之一成立, 则 $u = v \equiv 0$.

参考文献:

- [1] Smets D. Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities[J]. Transactions of the American Mathematical Society, 2005, 357(7):2909-2938.
- [2] Felli V, Terracini S. Elliptic equations with multi-singular inverse square potential and critical nonlinearity[J]. Communications in Partial Differential Equations, 2006, 31(3):469-495.
- [3] Dávila J, Pino M, Musso M, et al. Standing waves for supercritical nonlinear Schrödinger equations[J]. Journal of Differential Equations, 2007, 236(1):164-198.
- [4] Jeong W, Lee Y. Stable solutions and finite Morse index solutions of nonlinear elliptic equations with Hardy potential[J]. Nonlinear Analysis Theory Methods and Applications, 2013, 87(87):126-145.
- [5] Du Y H, Guo Z M. Finite Morse index solutions and asymptotics of weighted nonlinear elliptic equations[J]. Advances in Differential Equations, 2013, 18(7/8):737-768.
- [6] Gidas B, Spruck J. Global and local behavior of positive solutions of nonlinear elliptic equations[J]. Communications on Pure and Applied Mathematics, 1981, 34(4):525-598.
- [7] Gidas B, Spruck J. A priori bounds for positive solutions of nonlinear elliptic equations[J]. Communications in Partial Differential Equations, 1981, 6(8):883-901.
- [8] Dancer E N, Wei J C, Weth T. A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system[J]. Annales de l'Institut Henri Poincaré C, Analyse non linéaire, 2010, 27(3):953-969.
- [9] Polacik P, Quittner P, Souplet P. Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part I: Elliptic equations and systems[J]. Duke Mathematical Journal, 2007, 139(3):555-579.
- [10] Serrin J, Zou H H. Non-existence of positive solutions of Lane-Emden systems[J]. Differential Integral Equations, 1996, 9:635-653.
- [11] Souplet P. The proof of the Lane-Emden conjecture in four space dimensions[J]. Advances in Mathematics, 2009, 221(5):1409-1427.
- [12] 鲁亚南, 沃维丰. Levi 方程组的最优系统、对称约化和守恒律[J]. 宁波大学学报(理工版), 2018, 31(2):74-78.
- [13] Phan Q H, Souplet P. Liouville-type theorems and bounds of solutions of Hardy-Hénon equations[J]. Journal of Differential Equations, 2012, 252(3):2544-2562.
- [14] Fazly M, Ghoussoub N. On the Hénon-Lane-Emden conjecture[J]. Discrete and Continuous Dynamical Systems Series A, 2014, 34(6):2513-2533.
- [15] Yarur C S. A priori estimates for positive solutions for a class of semilinear elliptic systems[J]. Nonlinear Analysis, 1999, 36(1):71-90.
- [16] Mitidier E, Pohozaev S I. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities[J]. Proceedings of the Steklov Institute of Mathematics, 2001, 234:1-384.
- [17] Guedda M. Local and global nonexistence of solutions to nonlinear hyperbolic inequalities[J]. Applied Mathematics Letters, 2003, 16(4):493-499.

Liouville-type theorem of entire solutions for nonlinear Schrödinger system

WU Qian-qi, JI Ting, HU Liang-gen*

(Faculty of Science, Ningbo University, Ningbo, 315211, China)

Abstract: We take Hardy potential into consideration for the nonlinear Schrödinger system. Based on a combination of the variable substitutions transformations, the test function and the integral estimate, we complete the proof of Liouville-type theorem with the nonnegative classical entire solutions.

Key words: Schrödinger system; Liouville-type theorem; variable substitution; test function

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