

(2+1)维 Ito 方程的孤子解和有理解

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摘要: 基于 Hirota 双线性方法, 得到(2+1)维 Ito 方程的双线性形式, 由此可以求得(2+1)维 Ito 方程的 N-孤子解. 在二孤子解的基础上, 对参数取共轭, 可以得到一阶的呼吸子解; 再对二孤子解用长波极限和参数限制, 则可以得到 Ito 方程的一阶有理解.

关键词: Ito 方程; Hirota 双线性; 孤子解; 长波极限; 呼吸子解; 有理解

中图分类号: O175

文献标志码: A

文章编号: 1001-5132 (2019) 04-0063-05

众所周知, 非线性偏微分方程的精确解在许多复杂的物理现象和其他非线性工程问题的研究中起着至关重要的作用^[1-5]. 构造非线性发展方程的精确解在科学和工程应用中具有重要意义. 为了求解非线性发展方程的解, 检验这些解的物理性质, 现已有了许多非常有效的方法, 例如达布变换^[6-7]、Hirota 双线性方法^[8-9]、反散射方法^[10-11]、齐次平衡法^[12]、李群^[13]、贝克隆变换^[8]以及 KP 约化^[14-15]等.

Ito 方程是 19 世纪 80 年代数学家 Ito 通过用双线性的 KdV 方程的广义形式推导所得, 至今 Ito 方程的许多性质都已得到研究, 例如 Wang 等^[16]利用拓展同宿试验法得到了(2+1)维 Ito 方程的同宿呼吸子波、怪波和孤波文; Yang 等^[17]基于 Hirota 双线性研究了(2+1)维 Ito 方程的 lump 和 lump-孤子解; Zhang 等^[18]基于 Hirota 双线性得到了(2+1)维 Ito 方程的类 N-孤子解等.

本文应用 Hirota 双线性方法得到(2+1)维 Ito 方程的 N-孤子解; 进而基于长波极限理论和参数约束, 得到(2+1)维 Ito 方程的呼吸子解和 lump 解.

1 双线性形式和 N-孤子解

考虑如下形式的(2+1)维 Ito 方程:

$$u_{tt} + u_{xxx} + 6u_x u_t + 3u u_{xt} + 3u_{xx} \left(\int u dx \right)_t +$$

$$\alpha u_{yt} + \beta u_{xt} = 0, \quad (1)$$

其中: $u = u(x, y, t)$, α 与 β 是常数. 如果 $\alpha = 0$, $\beta = 0$, 方程(1)约化为(1+1)维的 Ito 方程. 作变换

$$u = 2(\ln f)_{xx}, \quad (2)$$

方程(1)化为:

$$\begin{aligned} & (D_t^2 + D_x^3 D_t + \alpha D_y D_t + \beta D_x D_t) f \cdot f = \\ & 2[f_{xt} f - f_t^2 + f_{xxx} - 3f_{xxt} f_x + \\ & 3f_{xt} f_{xx} - f_{xxx} f_t + \alpha(f_{yt} f - f_y f_t) + \\ & \beta(f_{xt} f - f_x f_t)] = 0, \end{aligned} \quad (3)$$

方程(3)中 f 是关于 x, y, t 的实函数, $D_t^2, D_x^3 D_t, D_y D_t, D_x D_t$ 是如下定义的 Hirota 双线性微分算子:

$$\begin{aligned} & D_t^m D_x^n f \cdot g = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n \cdot \\ & f(t, x) g(t', x')|_{t=t', x=x'}. \end{aligned} \quad (4)$$

下面通过用 Hirota 双线方法构造(2+1)维 Ito 方程的孤子解.

首先构造二孤子解, 假设方程(2)中 f 有如下形式:

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2, \quad (5)$$

其中:

$$\begin{aligned} & f_1 = e^{\eta_1} + e^{\eta_2}, \quad f_2 = e^{\eta_1 + \eta_2 + A_{12}}, \\ & \eta_s = k_s x + \omega_s t + p_s y + \varphi_s, \quad s = 1, 2, \end{aligned} \quad (6)$$

k_s, ω_s, ϕ_s 和 p_s 是任意的复数. 把方程(5)代入方程(3), 并比较 ε 的各次幂系数, 得如下方程组:

$$\begin{aligned}\varepsilon^0: & (D_t^2 + D_x^3 D_t + \alpha D_y D_t + \beta D_x D_t) \cdot \\ & (1 \cdot 1 + 1 \cdot 1) = 0, \\ \varepsilon^1: & (D_t^2 + D_x^3 D_t + \alpha D_y D_t + \beta D_x D_t) \cdot \\ & (1 \cdot f_1 + f_1 \cdot 1) = 0, \\ \varepsilon^2: & (D_t^2 + D_x^3 D_t + \alpha D_y D_t + \beta D_x D_t) \cdot \\ & (1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1) = 0.\end{aligned}\quad (7)$$

利用 Hirota 双线性微分算子式(4), 方程(7)就变为:

$$\begin{aligned}f_{1tt} + f_{1xxx} + \alpha f_{1yt} + \beta f_{1xt} &= 0, \\ \alpha f_1 f_{1yt} - \alpha f_{1t} f_{1y} + \beta f_1 f_{1xt} - \beta f_{1t} f_{1x} - \\ f_{1t}^2 + \alpha f_{2yt} + \beta f_{2xt} + f_1 f_{1tt} + f_1 f_{1xxx} - \\ f_{1t} f_{1xxx} - 3 f_{1x} f_{1xt} + 3 f_{1xx} f_{1xt} + \\ f_{2xxx} + f_{2tt} &= 0.\end{aligned}\quad (8)$$

把方程(6)中 f_1 代入方程(8)得:

$$\begin{aligned}e^{i\omega_1 + xk_1 + y p_1 + \phi_1} (k_1^3 \omega_1 + \alpha p_1 \omega_1 + \beta k_1 \omega_1 + \omega_1^2) + \\ e^{i\omega_2 + xk_2 + y p_2 + \phi_2} (k_2^3 \omega_2 + \alpha p_2 \omega_2 + \\ \beta k_2 \omega_2 + \omega_2^2) &= 0.\end{aligned}\quad (10)$$

把方程(6)中的 f_1 和 f_2 代入方程(9)得:

$$\begin{aligned}e^{(k_1 + k_2)x + (\omega_1 + \omega_2)t + (p_1 + p_2)y + (\phi_1 + \phi_2)} [(k_1 - k_2)^3 \cdot \\ (\omega_1 - \omega_2) + \alpha(\omega_1 - \omega_2)(p_1 - p_2) + \\ \beta(\omega_1 - \omega_2)(k_1 - k_2) + (\omega_1 - \omega_2)^2 + \\ e^{A_{12}} ((k_1 + k_2)^3 (\omega_1 + \omega_2) + \alpha(\omega_1 + \omega_2) \cdot \\ (p_1 + p_2) + \beta(\omega_1 + \omega_2)(k_1 + k_2) + \\ (\omega_1 + \omega_2)^2)] &= 0.\end{aligned}\quad (11)$$

通过解方程(10)和(11), 可以得到以下公式:

$$\begin{aligned}e^{A_{12}} = -[(k_1 - k_2)^3 (\omega_1 - \omega_2) + \alpha(\omega_1 - \omega_2)(p_1 - p_2) + \\ \beta(\omega_1 - \omega_2)(k_1 - k_2) + (\omega_1 - \omega_2)^2] / \\ [(k_1 + k_2)^3 (\omega_1 + \omega_2) + \alpha(\omega_1 + \omega_2)(p_1 + p_2) + \\ \beta(\omega_1 + \omega_2)(k_1 + k_2) + (\omega_1 + \omega_2)^2]\end{aligned}\quad (12)$$

和

$$k_s^3 \omega_s + \alpha p_s \omega_s + \beta k_s \omega_s + \omega_s^2 = 0, \quad s=1, 2. \quad (13)$$

当方程(5)中的 $\varepsilon=1$ 时, 则(2+1)维 Ito 方程的

二孤子解为:

$$u = 2 \ln(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}})_{xx}, \quad (14)$$

其中: $\eta_1, \eta_2, e^{A_{12}}$ 由方程(12)和(13)确定.

采用和二孤子解相同的步骤, 假设 f 有如下形式:

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots + \varepsilon^n f_n. \quad (15)$$

类似的, 把方程(15)代入方程(3)中, 并比较 ε 不同次幂的系数, 得到多个不同方程. 解这些方程, 可以得到方程(15)中 f 的表达式:

$$f = \sum_{\mu=0,1} \exp\left(\sum_{j<s} \mu_j \mu_s A_{js} + \sum_{j=1}^N \mu_j \eta_j\right), \quad (16)$$

其中, $\eta_j = k_j x + \omega_j t + p_j y + \phi_j$,

$$\begin{aligned}e^{A_{js}} = -[(k_j - k_s)^3 (\omega_j - \omega_s) + \alpha(\omega_j - \omega_s)(p_j - p_s) + \\ \beta(\omega_j - \omega_s)(k_j - k_s) + (\omega_j - \omega_s)^2] / \\ [(k_j + k_s)^3 (\omega_j + \omega_s) + \alpha(\omega_j + \omega_s)(p_j + p_s) + \\ \beta(\omega_j + \omega_s)(k_j + k_s) + (\omega_j + \omega_s)^2],\end{aligned}\quad (17)$$

这里

$$k_s^3 \omega_s + \alpha p_s \omega_s + \beta k_s \omega_s + \omega_s^2 = 0, \quad s=1, \dots, N. \quad (18)$$

在式(17)中 k_s, ω_s, p_s 和 ϕ_s 是任意复数. 在式(16)中, 记 $\sum_{\mu=0,1}$ 表示 $u_j = 0, 1 (j=1, 2, \dots)$ 所有可能组合的求和, 记 $\sum_{j<s}^N$ 表示当 $j < s$ 时 $\mu_j \mu_s A_{js}$ 所有可能组合的求和.

2 一阶呼吸解

在二孤子解式(14)中取 $\eta_1 = \eta_2^*$, 这时得到的解即为呼吸子解. 为了得到(2+1)维 Ito 方程的呼吸子解, 令参数之间的关系如下:

$$k_2^* = k_1, \quad p_2^* = p_1, \quad \omega_2^* = \omega_1, \quad \phi_2^* = \phi_1, \quad (19)$$

简单地说, 即是:

$$\begin{aligned}k_1 = k_{11} + ik_{12}, \quad k_2 = k_{11} - ik_{12}, \quad p_1 = p_{11} + ip_{12}, \\ p_2 = p_{11} - ip_{12}, \quad \omega_1 = \omega_{11} + i\omega_{12}, \\ \omega_2 = \omega_{11} - i\omega_{12}, \quad \phi_1 = \phi_2 = \phi_0,\end{aligned}\quad (20)$$

则方程(5)中的 f 可写成:

$$f = 2e^{\Theta_1} (\sqrt{M} \cosh(\Theta_1) + \cos(\Theta_2)), \quad (21)$$

其中:

$$\Theta_1 = k_{11}x + p_{11}y + \omega_{11}t + \phi_0 + \ln \sqrt{M},$$

$$\begin{aligned}\Theta_2 &= k_{12}x + p_{12}y + \omega_{12}t, M = e^{A_{12}}, \\ \Theta_3 &= k_{11}x + p_{11}y + \omega_{11}t + \varphi_0,\end{aligned}\quad (22)$$

式(22)中 $k_{11}, k_{12}, p_{11}, p_{12}, \omega_{11}, \omega_{12}, \varphi_0$ 都是实数.

再通过变换 $u = 2\ln(f)_{xx}$ 就可得到 Ito 方程的一阶呼吸子解.

选取适当的参数, 令条件(20)中

$$\begin{aligned}k_1 &= 1/2 + i, k_2 = 1/2 - i, p_1 = p_2 = 1/2, \\ \varphi_1 &= \varphi_2 = 0,\end{aligned}\quad (23)$$

此时表达式(21)中的函数 f 可写为:

$$\begin{aligned}f &= 4e^{(3/4)t+x+y} \cosh((3/4)t + x + y + \ln 2) + \\ &2e^{(3/8)t+(1/2)x+(1/2)y} \cos(x - (3/4)t).\end{aligned}\quad (24)$$

通过变换 $u = 2\ln(f)_{xx}$ 就得到了(2+1)维 Ito 方程的一阶呼吸子解.

从图 1~4 可以看出, (2+1)维 Ito 方程的一阶呼吸子解在 (x, y) 平面内是沿着 x 轴和 y 轴所成角的角平分线方向传播, 周期是 2π , 且在传播过程中图像的最高点不随时间 t 的变化而变化.

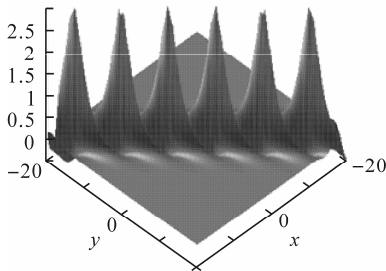


图 1 $t = 0$

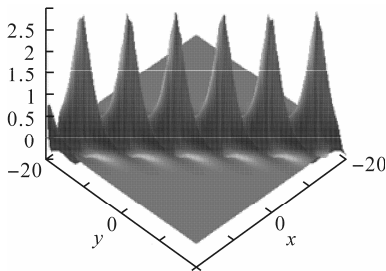


图 2 $t = -1$

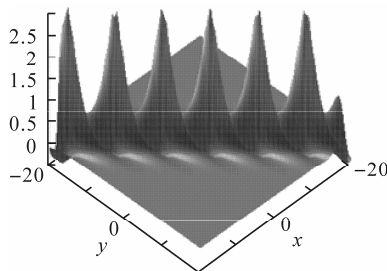


图 3 $t = 1$

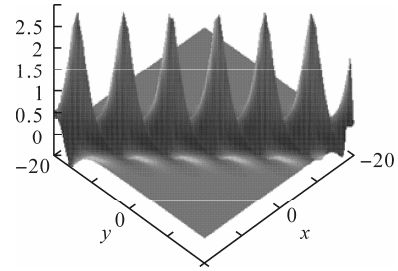


图 4 $t = -5$

3 一阶有理解

为了得到有理解, 对表达式(21)中的 f 用长波极限(令 $p_1, k_1 \rightarrow 0$), 且令 $e^{\varphi_0} = -1$, 则 f 关于 p_1, k_1 的泰勒展开为:

$$\begin{aligned}f &= \frac{[2\tilde{k}_1\tilde{k}_2x + (\tilde{k}_1\tilde{p}_2 + \tilde{k}_2\tilde{p}_1)y]^2}{4\tilde{k}_1\tilde{k}_2} - \\ &\frac{(\tilde{k}_1\tilde{p}_2 - \tilde{k}_2\tilde{p}_1)^2(\alpha t - y)^2}{4\tilde{k}_1\tilde{k}_2} + \\ &\frac{4\tilde{k}_1^2\tilde{k}_2^2(\tilde{k}_1^2 + \tilde{k}_2^2)}{\alpha(\tilde{k}_1^2 - \tilde{k}_2^2)(\tilde{k}_1\tilde{p}_2 - \tilde{k}_2\tilde{p}_1)},\end{aligned}\quad (25)$$

其中

$$\beta = -\frac{1}{2} \frac{\alpha(\tilde{k}_1\tilde{p}_2 + \tilde{k}_2\tilde{p}_1)}{\tilde{k}_1\tilde{k}_2},$$

$$k_1 = \tilde{k}_2^*, p_1 = \tilde{p}_2^*.$$

为了保证函数 f 是实函数, 且是正的, 参数之间还必须满足下面条件:

$$\begin{aligned}\frac{\tilde{k}_1^2 + \tilde{k}_2^2}{\alpha(\tilde{k}_1^2 - \tilde{k}_2^2)(\tilde{k}_1\tilde{p}_2 - \tilde{k}_2\tilde{p}_1)} &> 0 \Leftrightarrow \\ \frac{k_{11}^2 - k_{12}^2}{\alpha k_{11}k_{12}(k_{11}p_{12} - k_{12}p_{11})} &> 0.\end{aligned}\quad (26)$$

满足式(26)得到的有理解 u 是非退化的, 而且沿着

$$x(t) = -\frac{\alpha t(\tilde{k}_1\tilde{p}_2 + \tilde{k}_2\tilde{p}_1)}{2\tilde{k}_1\tilde{k}_2}, y(t) = \alpha t,$$

轨线保持不变. 无论何时, 当 $(x, y) \rightarrow \infty$ 时, $u \rightarrow 0$. 因此, 这个有理解是一个稳定的 lump 解, 而且有 3 个极值点:

$$A_1 = (-\frac{\alpha t(\tilde{k}_1\tilde{p}_2 + \tilde{k}_2\tilde{p}_1)}{2\tilde{k}_1\tilde{k}_2}, \alpha t),$$

$$A_2 = (-\frac{\alpha t(\tilde{k}_1\tilde{p}_2 + \tilde{k}_2\tilde{p}_1)}{2\tilde{k}_1\tilde{k}_2} + 4[3\alpha\tilde{k}_1^3\tilde{k}_2^3,$$

$$\begin{aligned}
& (\tilde{k}_1 - \tilde{k}_2)(\tilde{k}_1 + \tilde{k}_2)(\tilde{k}_1^2 + \tilde{k}_2^2) \cdot \\
& (\tilde{k}_1 \tilde{p}_2 - \tilde{k}_2 \tilde{p}_1)]^{1/2}, \alpha t), \\
A_3 = & \left(\frac{\alpha t(\tilde{k}_1 \tilde{p}_2 + \tilde{k}_2 \tilde{p}_1)}{2\tilde{k}_1 \tilde{k}_2} + 4[3\alpha \tilde{k}_1^3 \tilde{k}_2^3 \cdot \right. \\
& (\tilde{k}_1 - \tilde{k}_2)(\tilde{k}_1 + \tilde{k}_2)(\tilde{k}_1^2 + \tilde{k}_2^2) \cdot \\
& \left. (\tilde{k}_1 \tilde{p}_2 - \tilde{k}_2 \tilde{p}_1)]^{1/2}, \alpha t) \right).
\end{aligned}$$

这3个极值点是通过解偏微分方程组 $\partial u / \partial x = 0$ 和 $\partial u / \partial y = 0$ 得到.

令 $\tilde{k}_1 = \tilde{k}_2^* = (1/4 + i)k$, $\tilde{p}_1 = \tilde{p}_2 = p$, 则表达式(25)中的 f 变为:

$$f = \frac{16}{17} p^2 (\alpha t - y)^2 + \frac{1}{172} (17kx + 4py)^2 + \frac{4335}{1024} \frac{k^3}{\alpha p}, \quad (27)$$

有理解最后的表达式为:

$$u = \frac{289k^2}{86f} - \frac{289(17kx + 4py)^2 k^2}{7396f^2}. \quad (28)$$

选取式(28)中的参数为 $k=1$, $p=2$, $\alpha=1$ 得到有理解的图.

图5和图6展示了(2+1)维 Ito 方程的一阶有理解随时间 t 变化的图像, 在传播过程中能量保持不变; 图7是一阶有理解的等高线图, 黑色阴影部分代表了一阶有理解的最小值处; 图8是一阶有理解的密度图.

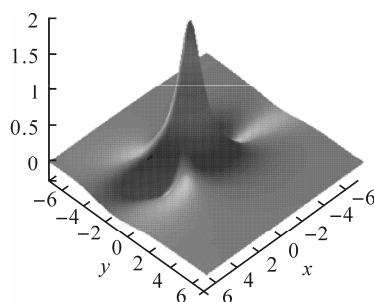


图5 $t=0$

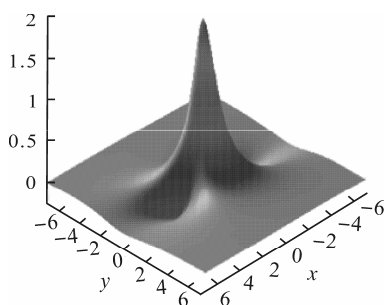


图6 $t=-1$

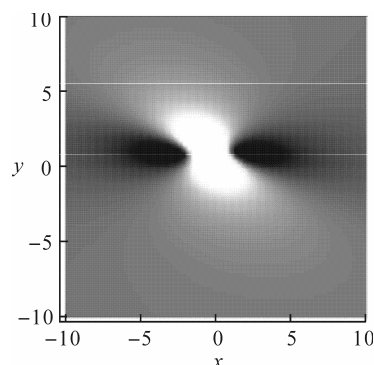


图7 $t=1$

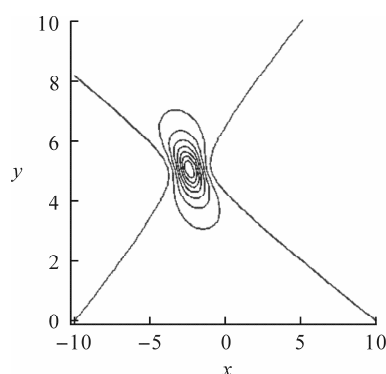


图8 $t=5$

4 结语

本文以软件 Maple 为工具, 以(2+1)维 Ito 方程为研究对象, 基于其双线性形式, 得到了 Ito 方程的 N-孤子解. 在二孤子解的基础上, 取共轭参数, 求得了一阶呼吸解(图 1~4). 进而运用长波极限理论和泰勒展, 从一阶呼吸解导出(2+1)维 Ito 方程的一阶有理解(图 5~8). 说明 Hirota 双线性方法和长波极限理论对于求解非线性发展方程的孤子解和有理解是一种直接且有效的方法. 这种方法也适用于研究数学问题和工程问题中的其他非线性模型方程.

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Soliton and rational solutions of the (2+1)-dimensional Ito equation

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Abstract: By employing the Hirota bilinear method, the bilinear form of the (2+1)-dimensional Ito equation can be derived. Based on the bilinear form, N -soliton solutions are constructed. On the condition of parameter conjugations, we can obtain breather solutions to the (2+1)-dimensional Ito equation from the two-soliton solution. By taking a long wave limit and making further parameter constraints, one-order rational solution to the (2+1)-dimensional Ito equation can be derived.

Key words: Ito equation; Hirota bilinear method; soliton solution; long-wave limit; breather solution; rational solution

(责任编辑 史小丽)