

A New Algorithm to Compute the Square Roots of r -circulant Matrices

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Abstract: A simplified form of r -circulant matrices is investigated, and a new algorithm to compute the square roots of r -circulant matrices is obtained using the properties of r -circulant matrices. In stead of calculating the eigenvalues of the r -circulant matrices, the proposed method only needs inter-matrix multiplication in its iterative sequence.

Key words: r -circulant matrices; primary square root; reduced form; Schur decomposition

CLC number: O515.21

Document code: A

Article ID: 1001-5132 (2012) 03-0035-04

The r -circulant matrices is a kind of important and special matrices which are applied in many areas of digital image processing, linear predicting, engineering calculation, math CAD. For recent years the properties and application of them have been extensively investigated.

Some scholars have given various algorithms for computing the square roots of r -circulant. In the [1], $A = C_r^n(a_0, a_1, a_2 \cdots a_{n-1})$ was given by the following formula:

$$A^{1/2} = C_r^n(b_0, b_1, \cdots, b_{n-1}), b_k = \frac{1}{n} \sum_{j=0}^{n-1} f(w_j)^{1/2} w_j^{-k},$$
$$k = 0, 1, \cdots, n-1,$$

where w_j is the root of $x^n = r$, $f(x) = \sum_{i=0}^{n-1} a_i x^i$, its

time complexity is $O(n \log 2n)$, unfortunately, the computational complexity of these algorithms are very amazing with the order of matrix increasing. The purpose of this paper is to obtain better results for computing the square roots of r -circulant matrices, on the basis of the reduced form of A , and an operation which can be done very efficiently on modern high

performance computers. This method is popularized from [2].

In the paper, we first construct the reduced forms of r -circulant matrices. Then we show that the problem of computing the square roots of a r -circulant matrix A can be reduced to that of computing the square roots of two half size matrices $B - \sqrt{r}C$ and $B + \sqrt{r}C$. On the basis of these discussions an efficient algorithm is presented to computer their square roots respectively. In the next section, we review some basic definition and notation. In section 3, we investigate the properties of r -circulant matrices, then discuss the form of square roots of r -circulant matrices and present an algorithm to solve the primary square root.

1 Preliminaries

Definition 1^[3] An $n \times n$ matrix $A = [a_{i,j}]_{i,j=1}^n$ is called a r -circulant matrix if it is the form of

$$a_{ij} = \begin{cases} a_{j-i}, & j \geq i, \\ ra_{n+j-i}, & j < i. \end{cases}$$

Received date: 2011-12-27.

JOURNAL OF NINGBO UNIVERSITY (NSEE): <http://3xb.nbu.edu.cn>

Foundation items: Supported by Zhejiang Provincial Natural Science Foundation (Y604089).

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Obviously the r -circulant matrix A is determined by parameter r and its first row elements $a_0, a_1, \dots, a_{n-1}$, then we denote $A = C_r^n(a_0, a_1, \dots, a_{n-1})$.

Especially let $r=1$, the matrix A is called a circulant matrix.

Lemma 1^[3] Let $A \in C^{n \times n}$ be a r -circulant matrix, then (i) If A is nonsingular, then A^{-1} is a r -circulant matrix. (ii) Let $B \in C^{n \times n}$ be an r -circulant matrix, then AB and $A \pm B$ are also r -circulant matrix, furthermore $AB=BA$.

For simplicity, we restrict the case of $n=2m$ and $r \neq 0$ in the following. Using the partition, the $n \times n$ r -circulant matrix A can be described as

$$A = \begin{pmatrix} B & C \\ rC & B \end{pmatrix}, \quad (1)$$

where B and C are $m \times m$ matrix.

Define

$$S = \begin{pmatrix} -1/\sqrt{r}I_m & 1/\sqrt{r}I_m \\ I_m & I_m \end{pmatrix}, \quad (2)$$

from S , we can obtain

$$S^{-1} = \begin{pmatrix} -\sqrt{r}/2I_m & 1/2I_m \\ \sqrt{r}/2I_m & 1/2I_m \end{pmatrix}, \quad (3)$$

where I_m is an m th unit matrix.

By using (1)–(3), we can obtain the following:

Lemma 2 If A is an r -circulant matrix, then

$$S^{-1}AS = \text{diag}(H, L), \quad (4)$$

where $H = B - \sqrt{r}C$ and $L = B + \sqrt{r}C$. We call the matrix of the right side of (4) the reduced form of the r -circulant matrix A .

Corollary 1 $r=1$, let

$$S = P = \frac{\sqrt{2}}{2} \begin{pmatrix} I_m & I_m \\ -I_m & I_m \end{pmatrix}, \quad S^{-1}AS = \text{diag}(H, L),$$

where $H=B-C, L=B+C$.

Proof It can be obtained from [2].

Definition 2 Matrix function via Hermite interpolation^[1]. Let g be defined on the spectrum of $A \in C^{n \times n}$ and let ψ be the minimal polynomial of A . Then $g(A) := P(A)$, where P is a polynomial of degree less than $\sum_{i=1}^s n_i = \deg \psi$ that satisfies the interpolation

conditions

$$P^{(j)}(\lambda_i) = g^{(j)}(\lambda_i), \\ j = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, s.$$

There is a unique such P and it is known as the Hermite interpolation polynomial.

Lemma 3^[2] Let the nonsingular matrix $A \in C^{n \times n}$ have the Jordan canonical form $Z^{-1}AZ = J = \text{diag}(J_1, J_2, \dots, J_p)$, and let $s \leq p$ be the number of distinct eigenvalues of A . Let

$$L_k^{(j)} = L_k^{(j)}(\lambda_k) = \begin{pmatrix} g(\lambda_k) & g'(\lambda_k) & \dots & \frac{g^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ & f(\lambda_k) & \ddots & \vdots \\ & & \ddots & g'(\lambda_k) \\ & & & g(\lambda_k) \end{pmatrix},$$

where $g(x) = x^{1/2}$, then A has precisely 2^s square roots that are primary functions of A , given by

$$X_j = Z \text{diag}(L_1^{(j_1)}, L_2^{(j_2)}, \dots, L_p^{(j_p)}) Z^{-1}, \quad j = 1: 2^s,$$

corresponding to all possible choices of $j_1, \dots, j_p, j_k = 1$ or 2 , subject to the constraint that $j_i = j_k$ whenever $\lambda_i = \lambda_k$.

If $s < p$, A has nonprimary square roots. They form parameterized families

$$X_j(U) = ZU \text{diag}(L_1^{(j_1)}, L_2^{(j_2)}, \dots, L_p^{(j_p)}) U^{-1} Z^{-1}, \\ j = 2^s + 1: 2^p,$$

where $j_k = 1$ or 2 , U is an arbitrary nonsingular matrix that commutes with J , and for each j there exist i and k , depending on j , such that $\lambda_i = \lambda_k$ while $j_i \neq j_k$.

2 The square root of r -circulant matrices

In this section we will discuss the square roots of r -circulant matrices.

Theorem 1 let $A \in C^{n \times n}$ be a nonsingular r -circulant matrix and let $X^2 = A$, where X are the primary function of A . Then all square roots X are r -circulant matrices.

Proof By assumption, $X^2 = A$ and $X = g(A)$, here $g(x) = x^{1/2}$ which are clearly defined on the

spectrum of A , including the case that the eigenvalues of the matrix A are the complex numbers. From Definition 2, we can construct a polynomial P such that $P(A) = g(A)$. In terms of Lemma 1, the polynomial $X = P(A)$ is an r -circulant matrix since A is an r -circulant matrix.

Theorem 2 Let $A \in C^{n \times n}$ be an r -circulant matrix. If A has an r -circulant square root, then each $B - \sqrt{r}C$ and $B + \sqrt{r}C$ in Lemma 2 has a square root, respectively.

Proof Note first that A has the reduced form (4). From hypothesis, denote an r -circulant square root of A by $\tilde{X}$. By Lemma 2, there holds

$$S^{-1} \tilde{X} S = \text{diag}(X_1, X_2), \quad (5)$$

where S is defined by the relation (2). In fact, $\tilde{X}^2 = A$, implies that $X_1^2 = B - \sqrt{r}C$ and $X_2^2 = B + \sqrt{r}C$ hold simultaneously, that is, X_1 and X_2 are square roots of $B - \sqrt{r}C$ and $B + \sqrt{r}C$, respectively.

Note that the relation (5) shows that the problem of computing the r -circulant square root of a r -circulant matrix $A \in C^{2m \times 2m}$ can be reduced to that of computing the square roots of two half size matrices $B - \sqrt{r}C$ and $B + \sqrt{r}C$, where $X_1^2 = B - \sqrt{r}C$ and $X_2^2 = B + \sqrt{r}C$.

By Lemma 3 and Theorem 1, we get that any nonsingular r -circulant matrix always has a circulant square root. Generally, a nonsingular r -circulant matrix A , besides the r -circulant square roots, possibly has other kinds of square roots, which are not primary functions of A . The existence and the families of square roots depend on the spectrums of H and L in (2). The following theorem gives a classification of the entire square roots of a nonsingular r -circulant matrix.

Theorem 3 Let the nonsingular r -circulant matrix $A \in R^{n \times n}$ have the reduced form (1). Assume that M has s distinct eigenvalues and N has t distinct eigenvalues. Let $M = Z_M J_M Z_M^{-1}$ with $J_M = \text{diag}(J_1, J_2, \dots, J_l)$ the Jordan decompositions of M , and let $N = Z_N J_N Z_N^{-1}$ with $J_N = \text{diag}(\tilde{J}_1, \tilde{J}_2, \dots, \tilde{J}_h)$ be the Jordan decompositions of N .

Let $|\sigma(M) \cap \sigma(N)| = \alpha$, where $\sigma(M)$ denotes

the spectrum of matrix M , then A has $2^{s+t-\alpha}$ r -circulant square roots that are primary function of A , given by

$$\hat{X} = PZ \hat{L} Z^{-1} P^{-1}, \quad (6)$$

with

$$Z = \text{diag}(Z_M, Z_N), \quad \hat{L} = \text{diag}(\hat{L}_M, \hat{L}_N), \quad (7)$$

where $\hat{L}_M$ denotes a square root of J_M , which is a primary function of J_M , and $\hat{L}_N$ denotes a square root of J_N , which is a primary function of J_N . If $s+t < l+h$, A has nonprimary square roots which form $2^{l+r} - 2^{s+t-\alpha}$ parameterized families given by

$$\hat{X}(U) = PZU \hat{L} U^{-1} Z^{-1} P^{-1}, \quad (8)$$

where $Z, \hat{L}$ are defined in (6) and U is an arbitrary matrix that commutes with $J = \text{diag}(J_M, J_N)$.

Proof According to the hypothesis $\text{diag}(M, N)$ has $s+t-\alpha$ distinct eigenvalues and $l+h$ Jordan blocks. Then by Lemma 3, we can get that A has $2^{s+t-\alpha}$ square roots which are a primary function of A and take the form (6). By Theorem 1, the square roots are r -circulant matrices. Using Lemma 3 again, we can get form (8)

Algorithm Compute a primary square root of a nonsingular r -circulant matrix $A \in C^{n \times n}$.

Step 1 Compute the reduced form $S^{-1}AS = \text{diag}(H, L)$ in (8).

Step 2 Compute the Schur decomposition $T_H = U_H^T H U_H$ and $T_L = U_L^T L U_L$, respectively, where T_H and T_L are two upper triangular matrices.

Step 3 Compute the upper triangular square roots $P_H = g(T_H)$ and $P_L = g(T_L)$, where P_H has $n/2$ distinct eigenvalues and so does P_L , here $g = \lambda^{1/2}$ is defined on $\lambda \in \sigma(\text{diag}(T_H, T_L))$.

Step 4 Compute $X_H = U_H P_H U_H^T$ and $X_L = U_L P_L U_L^T$.

Step 5 Obtain $X = S \text{diag}(X_H, X_L) S^T$.

The time complexity of calculating the circulant matrix square root is $15/2n^3 + O(n^2)$ in [2], and the change of time complexity caused by r can be ignored, so its time complexity is also $15/2n^3 + O(n^2)$.

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