

Dual of two-parameter B -valued martingales Hardy space ${}_p H_\alpha^s(B)$ with small index

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Abstract: By introducing a new type of Campanato spaces for two-parameter B -valued martingale and employing the classical technique of atomic decomposition, it is proved that the dual space of two-parameter B -valued martingale space ${}_p H_\alpha^s(B)$ defined by conditional p -variation function for small index $0 < \alpha \leq 1$ is equivalent to $L_{q,\beta}^{s(q)}(B^*)$. The results obtained in this work extends the corresponding known results derived from two-parameter real-valued martingale setting and one-parameter B -valued martingale setting to two-parameter B -valued martingale setting.

Key words: two-parameter B -valued martingale space; dual space; atomic decomposition

CLC number: O211.6

Document code: A

Article ID: 1001-5132 (2020) 02-0107-06

1 Introduction

Regarding the duality theory of martingale Hardy space ${}_p H_\alpha^s$ associated with the conditional p -variation function, in the setting of real-valued martingales, it was proved by Herz^[1] that the dual space of one-parameter martingale Hardy space ${}_2 H_1^s(R)$ is $BMO_2(R)$. In addition, under the conditions of an atomic σ -algebra sequence, Herz^[2] proved that the dual space of ${}_2 H_\alpha^s(R)$ in the case of $0 < \alpha \leq 1$ is equivalent to r -ordered squared Lipschitz space $L_{2,r}(R)$ with $r = 1/\alpha - 1$. Employing the technique of atomic decomposition, Weisz^[3] proved that the result is also true for an arbitrary σ -algebra sequence. Similar results of two-parameter martingale and two-parameter strong martingale Hardy space ${}_2 H_\alpha^s(R)$ were proved by Weisz^[4-5]. As for B -valued martingales setting, Yu^[6] studied the dual space of the one-parameter B -valued martingale Hardy space ${}_p H_\alpha^s(B)$ in the case of

$0 < \alpha \leq 1$. Ma et al^[7] investigated the dual space of the one-parameter B -valued martingale weak Hardy space $w_p H_\alpha^s(B)$ in the case of $0 < \alpha \leq 1$. But the problem about the dual of two-parameter B -valued martingale Hardy space ${}_p H_\alpha^s(B)$ for $0 < \alpha \leq 1$ still remains unresolved.

The main purpose of this paper is to characterize the dual of two-parameter B -valued martingales Hardy spaces ${}_p H_\alpha^s(B)$ for $0 < \alpha \leq 1$. The results extend the corresponding known theorems from two-parameter real-valued martingale setting and one-parameter B -valued martingale setting to two-parameter B -valued martingale setting.

2 Preliminaries and notations

Let (Ω, F, P) be a complete probability space and $(B, \|\cdot\|)$ denote a real separable Banach space with $\dim B \geq 2$. Let $\{F_n, n \in \mathbb{N}^2\}$ be a family of non-decreasing sub- σ -algebras of F with respect to the

收稿日期: 2018-12-18.

宁波大学学报(理工版)网址: <http://journal.lg.nbu.edu.cn/>

基金项目: 国家自然科学基金(61573235).

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partial ordering on N^2 such that $F = \sigma(\bigcup_{n \in N^2} F_n)$. Set $F_{n-1} = F_{n_1-1, n_2-1}$, the expectation operator is denoted by E , and conditional expectation operators with respect to F_n and F_{n-1} are denoted by E_n and E_{n-1} .

An integrable B -valued sequence of functions $f = (f_n, n \in N^2)$ is said to be a martingale if it is adapted to $\{F_n, n \in N^2\}$ and $E_n f_m = f_n$ for all $n \leq m$.

We define the sequence of martingale differences $df = (df_n, n \in N^2)$ as follows:

$$df_n = \begin{cases} 0, & n_1 n_2 = 0, \\ f_{n_1, n_2} - f_{n_1-1, n_2} - f_{n_1, n_2-1} + f_{n_1-1, n_2-1}, & n_1 n_2 \neq 0. \end{cases}$$

And set

$$F_{-1} = F_{-1, i} = F_{i, -1} = F_0 = \{\phi, \Omega\},$$

$$f_{-1} = f_{-1, i} = f_{i, -1} = f_0 = 0.$$

A function ν which maps Ω into the set of subspaces of $N^2 \cup \{\emptyset\}$ is said to be a two-parameter stopping time relative to $\{F_n, n \in N^2\}$ if for all $\omega \in \Omega$, $\nu(\omega)$ are incomparable and for any $n \in N^2$, $\{\omega \in \Omega : n \in \nu(\omega)\} \in F_n$. The set of all two-parameter stopping times will be denoted by T_2 . It is easy to see that $\nu \in T_2$ iff $\{\nu \leq n\} \in F_n$ for all $n \in N^2$. Furthermore, $\{\nu \leq n\} \in F_{n-1}$ ($n \in N^2$) if $\nu \in T_2$.

For a martingale $f = (f_n, n \in N^2)$ and stopping time ν , the stopped martingale $f^{(\nu)} = (f_n^{(\nu)}, n \in N^2)$ is defined by

$$f_n^{(\nu)} = \sum_{m \leq n} \chi(\nu \leq m) df_m, \quad n \in N^2.$$

For a two-parameter B -valued martingale $f = (f_n, n \in N^2)$ and $1 \leq p < \infty$ we denote by f^* , $s^{(p)}(f)$ and $\|f\|_p$ its maximal function, conditional p -variation function and L_p -norm as follows:

$$f_n^* = \sup_{i \leq n} \|f_i\|, \quad f^* = \sup_{n \in N^2} \|f_n\|,$$

$$s_n^{(p)}(f) = \left(\sum_{0 \leq i \leq n} E_{i-1} \|df_i\|^p \right)^{1/p},$$

$$s^{(p)}(f) = \sup_{n \in N^2} s_n^{(p)}(f),$$

$$\|f\|_p = \sup_{n \in N^2} \|f_n\|_p.$$

Then, for $0 < \alpha \leq \infty$, we define Hardy spaces of two-parameter B -valued martingales by

$$H_\alpha^*(B) = \{f : \|f\|_{H_\alpha^*(B)} = \|f^*\|_\alpha < \infty\},$$

$${}_p H_\alpha^s(B) = \{f : \|f\|_{{}_p H_\alpha^s(B)} = \|s^{(p)}(f)\|_\alpha < \infty\},$$

$$L_\alpha(B) = \{f : \|f\|_{L_\alpha(B)} = \sup_{n \in N^2} [E(\|f_n\|^\alpha)]^{1/\alpha} < \infty\}.$$

Similar to real-valued martingales, in the case of $\alpha \geq 1$, ${}_p H_\alpha^s(B)$ are Banach spaces, and in the case of $0 < \alpha \leq 1$, ${}_p H_\alpha^s(B)$ are quasi Banach spaces.

To characterize the duality of the spaces ${}_p H_\alpha^s(B)$, for any $1 < p < \infty$ and $\alpha \geq 0$ we define the Campanato spaces of two-parameter B -valued martingales by

$$L_{p, \alpha}(B) = \{f : \|f\|_{L_{p, \alpha}(B)} < \infty\},$$

$$L_{p, \alpha}^s(B) = \{f : \|f\|_{L_{p, \alpha}^s(B)} < \infty\},$$

with

$$\|f\|_{L_{p, \alpha}(B)} = \sup_{\nu \in T_2} P(\nu \neq \infty)^{-\frac{1}{p} - \alpha} \|f - f^{(\nu)}\|_{L_p(B)},$$

$$\|f\|_{L_{p, \alpha}^s(B)} = \sup_{\nu \in T_2} P(\nu \neq \infty)^{-\frac{1}{p} - \alpha} \|s^{(p)}(f - f^{(\nu)})\|_{L_p(B)},$$

where we also define the BMO spaces by

$$BMO_\alpha(B) = \{f : \|f\|_{BMO_\alpha(B)} = \sup_n \|E_n(\|f - f_n\|^\alpha)^{1/\alpha}\|_\infty < \infty\};$$

$${}_p BMO_\alpha^s(B) = \{f : \|f\|_{{}_p BMO_\alpha^s(B)} = \sup_n \|E_n[\|s^{(p)}(f - f_n)\|^\alpha]^{1/\alpha}\|_\infty < \infty\}.$$

For any Banach space B , it is easy to prove that $L_{p, 0}(B) = BMO_p(B)$ and $L_{p, 0}^s(B) = {}_p BMO_p^s(B)$, the results are similar to real-valued martingales.

Two normed or quasi-normed spaces X and Y are equivalent (or $X \approx Y$ and $\|\cdot\|_X = \|\cdot\|_Y$ in symbols) if and only if X and Y are isomorphic and their norms or quasi-norms are equivalent, i.e. there exist constants $C_1, C_2 > 0$, such that for all $f \in X$,

$$C_1 \|f\|_X \leq \|f\|_Y \leq C_2 \|f\|_X.$$

The dual of a normed or quasi-normed space X is denoted by X^* , we call Y the dual space of X if $X^* \approx Y$.

In this paper, we use C for the constants which can have different values in different places.

3 Dual space

In this section, we will describe the dual of two-parameter B -valued martingale Hardy space ${}_p H_\alpha^s(B)$ for small index $0 < \alpha \leq 1$.

Definition 1^[8] A two-parameter B -valued martingale $a=(a_n, n \in N^2)$ is called an atomic martingale of the type $A^{s(p)}(\alpha, \infty)$ ($1 < p < \infty$, $0 < \alpha < \infty$), if there is a corresponding stopping time $\nu \in T_2$, such that

- (i) $a_n = 0$, if $\nu \ll n, \forall n \geq 0$;
- (ii) $\|s^{(p)}(a)\|_\infty \leq P(\nu \neq \infty)^{-1/\alpha}$.

Lemma 1^[8] Let $1 < p < +\infty$, $0 < \alpha \leq p$, B is a Banach space. Then, for every two-parameter B -valued martingale $f=(f_n, n \in N^2) \in {}_p H_\alpha^s(B)$, there exists a sequence of type $A^{s(p)}(\alpha, \infty)$ atomic martingales $\{a^{(k)} = (a_n^{(k)}, n \in N^2)\}_{k \in \mathbb{Z}}$ with the corresponding stopping time ν_k and a sequence of positive numbers $\{\mu_k\}_{k \in \mathbb{Z}} \in l_\alpha$, such that for all $n \in N^2$

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^{(k)}, \text{ a.e.,}$$

and

$$\|f\|_{{}_p H_\alpha^s(B)} \approx \inf \left(\sum_{k \in \mathbb{Z}} \mu_k^\alpha \right)^{1/\alpha} < \infty,$$

where the infimum is taken over all such decompositions of f of the form (1). Furthermore, all of the atomic martingales $\{a^{(k)} = (a_n^{(k)}, n \in N^2)\}_{k \in \mathbb{Z}}$ belong to the space ${}_p H_\alpha^s(B)$ too.

Lemma 2 Let B be a Banach space, $1 < p < +\infty$, $0 < \alpha \leq 1$, then ${}_p H_p^s(B)$ is dense in ${}_p H_\alpha^s(B)$.

Proof For any two-parameter B -valued martingale $f=(f_n, n \in N^2) \in {}_p H_\alpha^s(B)$, by lemma 1, there is a sequence $\{a^{(k)} = (a_n^{(k)}, n \in N^2)\}_{k \in \mathbb{Z}}$ of atomic martingales of type $A^{s(p)}(\alpha, \infty)$, such that for all $n \in N^2$,

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^{(k)} = \sum_{k \in \mathbb{Z}} (f_n^{(\nu_{k+1})} - f_n^{(\nu_k)}), \text{ a.e.}$$

For any natural number N , define martingale $g^{(N)} = (g_n^{(N)}, n \in N^2)$, such that for all $n \in N^2$ and $g_n^{(N)} = \sum_{k=-N}^N \mu_k a_n^{(k)}$.

On one hand, since $s^{(p)}(f_n^{(\nu_k)}) \leq 2^k$, we have

$$\begin{aligned} s^{(p)}(g_n^{(N)}) &= s^{(p)}\left(\sum_{k=-N}^N \mu_k a_n^{(k)}\right) = s^{(p)}\left(\sum_{k=-N}^N f_n^{(\nu_{k+1})} - f_n^{(\nu_k)}\right) \\ f_n^{(\nu_k)} &= s^{(p)}(f_n^{(\nu_{N+1})} - f_n^{(\nu_{-N})}) \leq s^{(p)}(f_n^{(\nu_{N+1})}) + \\ s^{(p)}(f_n^{(\nu_{-N})}) &\leq 2^{N+1} + 2^{-N} < +\infty, \forall n \in N^2. \end{aligned}$$

Then $\|s^{(p)}(g^{(N)})\|_{L_p(B)} \leq 2^{N+1} + 2^{-N} < +\infty$. Therefore, $g^{(N)} \in {}_p H_p^s(B)$.

On the other hand, for any $n \in N^2$, we have $f_n -$

$$g_n^{(N)} = f_n - \sum_{k=-N}^N \mu_k a_n^{(k)} = (f_n - f_n^{(\nu_{N+1})}) + f_n^{(\nu_{-N})}, \text{ and}$$

$$\begin{aligned} s_n^{(p)}(f_n - f_n^{(\nu_{N+1})})^p &= \sum_{i=1}^n E_{i-1}(\|d_i f\|^p) - \\ \sum_{i=1}^{n \wedge \nu_{N+1}} E_{i-1}(\|d_i f\|^p) &= s_n^{(p)}(f)^p - s_{n \wedge \nu_{N+1}}^{(p)}(f)^p. \end{aligned} \quad (1)$$

Because $\nu_{N+1} \rightarrow +\infty$, a.e. as $N \rightarrow +\infty$, so $s_n^{(p)}(f_n - f_n^{(\nu_{N+1})}) \rightarrow 0$, a.e. as $N \rightarrow +\infty$. From (1), we find that $s_n^{(p)}(f_n - f_n^{(\nu_{N+1})})^\alpha$ can be majorized by integrable function $s_n^{(p)}(f)^\alpha$, and $s^{(p)}(f^{(\nu_{-N})}) \leq 2^{-N} \rightarrow 0$ as $N \rightarrow +\infty$, we conclude that the series $\sum_{k=-N}^N \mu_k a_n^{(k)}$ converges to f_n in ${}_p H_\alpha^s(B)$ quasi-norm. That is to say, the martingale sequence $\{g^{(N)}\}_{N \geq 1}$ converges to f in ${}_p H_\alpha^s(B)$.

All in all, it is shown that ${}_p H_p^s(B)$ is dense in ${}_p H_\alpha^s(B)$.

Lemma 2 is proved.

Theorem 1 Let B be a Banach space, $1 < p < +\infty$, $0 < \alpha \leq 1$, then $L_{q,\beta}^{s(q)}(B^*) \subset ({}_p H_\alpha^s(B))^*$, where $q = p/(p-1)$ and $\beta = 1/\alpha - 1$.

Proof By lemma 2, ${}_p H_p^s(B)$ is dense in ${}_p H_\alpha^s(B)$. For any two-parameter B -valued martingale $\phi = (\phi_n, n \in N^2) \in L_{q,\beta}^{s(q)}(B^*)$, $s^{(q)}(\phi) \in L_q(R)$; and for any two-parameter B -valued martingale $f = (f_n, n \in N^2) \in {}_p H_p^s(B)$, $s^{(p)}(f) \in L_p(R)$. Since for any $i, j \in N^2$, $d_i f \in L_p(B)$ and $d_j \phi \in L_q(B^*)$, then $d_i f \cdot d_j \phi \triangleq \langle d_i f, d_j \phi \rangle$ makes sense and satisfies

$$\begin{aligned} |E(d_i f \cdot d_j \phi)| &= |E(\langle d_i f, d_j \phi \rangle)| \leq \\ \|d_i f\|_{L_p(B)} \cdot \|d_j \phi\|_{L_q(B^*)} &< +\infty. \end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned} |E(f_n \cdot \phi_n)| &= |E(\sum_{i=1}^n d_i f \cdot \sum_{j=1}^n d_j \phi)| = \\ |E(\sum_{i=1}^n d_i f \cdot d_i \phi)| &= |E[\sum_{i=1}^n E_{i-1}(d_i f \cdot d_i \phi)]| \leq \\ E[(\sum_{i=1}^n (E_{i-1}(\|d_i f\|^p))^{\frac{1}{p}} \cdot (E_{i-1}(\|d_i \phi\|^q))^{\frac{1}{q}})] &\leq \\ E[(\sum_{i=1}^n E_{i-1}(\|d_i f\|^p))^{\frac{1}{p}} \cdot (\sum_{i=1}^n E_{i-1}(\|d_i \phi\|^q))^{\frac{1}{q}}] &= \\ E(s_n^{(p)}(f) s_n^{(q)}(\phi)) &\leq \|s_n^{(p)}(f)\|_{L_p(R)} \|s_n^{(q)}(\phi)\|_{L_q(R)} \leq \\ \|s^{(p)}(f)\|_{L_p(R)} \|s^{(q)}(\phi)\|_{L_q(R)} &< +\infty, \forall n \geq 1. \end{aligned} \quad (2)$$

Therefore, we can define a bounded linear functional $l_\phi(\cdot)$ on ${}_p H_\alpha^s(B)$ as bellow

$$l_\phi(f) \triangleq E(f \phi) \triangleq \lim_{n \rightarrow \infty} E(f_n \phi_n), \forall f \in {}_p H_p^s(B). \quad (3)$$

On the other hand, for any two-parameter B -valued martingale $f = (f_n, n \in \mathbb{N}^2) \in {}_p H_p^s(B)$, take the same stopping times ν_k , atomic martingales $a^{(k)}$ and a sequence of positive numbers $\mu_k (k \in \mathbb{Z})$ as we did in lemma 1. Then, for any $n \geq 1$, we have

$$E(f\phi) = \sum_{k \in \mathbb{Z}} \mu_k E(a_n^{(k)} \phi_n). \quad (4)$$

By a similar derivation as in (2), we also have

$$\begin{aligned} |E(a_n^{(k)} \phi_n)| &= |E[a_n^{(k)}(\phi_n - \phi_n^{\nu_k})]| = |E[\sum_{i=1}^n d_i a^{(k)}]| \\ &= |\sum_{j=1}^n d_j(\phi - \phi^{\nu_k})| = E[\sum_{i=1}^n d_i a^{(k)} d_i(\phi - \phi^{\nu_k})] \leq \\ &= E[(\sum_{i=1}^n E_{i-1}(\|d_i a^{(k)}\|^p))^{1/p} \cdot \\ &= (\sum_{i=1}^n E_{i-1}(\|d_j(\phi - \phi^{\nu_k})\|^q))^{1/q}] = \\ &= E[s_n^{(p)}(a^{(k)}) s_n^{(q)}(\phi - \phi^{\nu_k})] \leq \\ &= \|s_n^{(p)}(a^{(k)})\|_{L_p(R)} \|s_n^{(q)}(\phi - \phi^{\nu_k})\|_{L_q(R)}. \end{aligned} \quad (5)$$

From (4) and (5), we get

$$\begin{aligned} |E(f_n \phi_n)| &\leq \sum_{k \in \mathbb{Z}} |\mu_k| \cdot \|s_n^{(p)}(a^{(k)})\|_{L_p(R)} \cdot \|s_n^{(q)}(\phi - \phi^{\nu_k})\|_{L_q(R)} \\ &= \sum_{k \in \mathbb{Z}} |\mu_k| \cdot [E(s_n^{(p)}(a^{(k)})^p \chi_{\{\nu_k \neq \infty\}})]^{1/p} \cdot \\ &= \|s_n^{(q)}(\phi - \phi^{\nu_k})\|_{L_q(R)} \leq \sum_{k \in \mathbb{Z}} |\mu_k| \cdot [P(\nu_k \neq \infty)]^{-\frac{p}{\alpha}} \cdot \\ &= P(\nu_k \neq \infty)^{\frac{1}{p}} \cdot \|s_n^{(q)}(\phi - \phi^{\nu_k})\|_{L_q(R)} = \\ &= \sum_{k \in \mathbb{Z}} |\mu_k| \cdot [P(\nu_k \neq \infty)]^{\frac{1}{p} - \frac{1}{\alpha}} \cdot \|s_n^{(q)}(\phi - \phi^{\nu_k})\|_{L_q(R)} = \\ &= \sum_{k \in \mathbb{Z}} |\mu_k| \cdot [P(\nu_k \neq \infty)]^{-\frac{1}{q}(\frac{1}{\alpha} - 1)} \cdot \|s_n^{(q)}(\phi - \phi^{\nu_k})\|_{L_q(R)} = \\ &= \sum_{k \in \mathbb{Z}} |\mu_k| \cdot [P(\nu_k \neq \infty)]^{-\frac{1}{q} - \beta} \cdot \|s_n^{(q)}(\phi - \phi^{\nu_k})\|_{L_q(R)} \leq \\ &= \sum_{k \in \mathbb{Z}} |\mu_k| \cdot \|\phi\|_{L_{q,\beta}^{s(q)}(B^*)}. \end{aligned} \quad (6)$$

Since $0 < \alpha \leq 1$, by (3) and (6), we have

$$|l_\phi(f)|^\alpha \leq \sum_{k \in \mathbb{Z}} |\mu_k|^\alpha \cdot \|\phi\|_{L_{q,\beta}^{s(q)}(B^*)}^\alpha.$$

In other words,

$$|l_\phi(f)| \leq (\sum_{k \in \mathbb{Z}} |\mu_k|^\alpha)^{1/\alpha} \cdot \|\phi\|_{L_{q,\beta}^{s(q)}(B^*)}.$$

That is,

$$|l_\phi(f)| \leq \inf_{k \in \mathbb{Z}} (\sum_{k \in \mathbb{Z}} |\mu_k|^\alpha)^{1/\alpha} \cdot \|\phi\|_{L_{q,\beta}^{s(q)}(B^*)} \leq$$

$$C \|f\|_{{}_p H_p^s(B)} \cdot \|\phi\|_{L_{q,\beta}^{s(q)}(B^*)},$$

which demonstrates that the bounded linear functional $l_\phi(\cdot)$ can be extended from ${}_p H_p^s(B)$ onto ${}_p H_p^s(B)$ keeping its norm. This implies that $L_{q,\beta}^{s(q)}(B^*) \subset ({}_p H_p^s(B))^*$.

Theorem 1 is proved.

Now, to show that each bounded linear functional $l(\cdot)$ in $({}_p H_p^s(B))^*$ comes from some $\phi \in L_{q,\beta}^{s(q)}(B^*)$ as $l_\phi(\cdot)$ defined as in theorem 1, a type of mixed spaces must be introduced.

Let $L_\alpha(\tilde{l}_p(B))$ denote the spaces of all $\theta = (\theta_1, \theta_2, \dots)$, where $\theta_i (i = 1, 2, 3, \dots)$ are two-parameter B -valued random variables such that each $\theta \in L_\alpha(\tilde{l}_p(B))$ satisfies

$$\|\theta\|_{L_\alpha(\tilde{l}_p(B))} \triangleq (\sum_{i=1}^\infty E_{i-1}(\|\theta_i\|^p))^{1/p} \|_{L_\alpha(R)} < +\infty.$$

Let us repeat the usual space $L_\alpha(l_p(B))$ in Ref. [9], which is the class of all sequences $\theta = (\theta_1, \theta_2, \dots)$ such that each $\theta \in L_\alpha(l_p(B))$ satisfies

$$\|\theta\|_{L_\alpha(l_p(B))} = (\sum_{i=1}^\infty \|\theta_i\|^p)^{1/p} \|_{L_\alpha(R)} < +\infty.$$

Certainly, there exist significant differences between the space $L_\alpha(\tilde{l}_p(B))$ and the space $L_\alpha(l_p(B))$. Nevertheless, as a special case, when $\alpha = p$, the space $L_p(\tilde{l}_p(B))$ is just the usual space $L_p(l_p(B))$. If B is a reflexive Banach space, then the dual space of $L_p(l_p(B))$ is $L_q(l_q(B^*))$, where $1/p + 1/q = 1$. So, similar with lemma 4 in Ref. [9], we can get the following lemma immediately.

Lemma 3 Let B be a reflexive Banach space, $1 < p < +\infty$. Then, for any bounded linear functional $l(\cdot)$ on $L_p(\tilde{l}_p(B))$, there exists some $\xi \in L_q(\tilde{l}_q(B^*))$ such that $l(\theta) = \sum_{i=1}^\infty E(\xi_i \theta_i)$, $\forall \theta \in L_p(\tilde{l}_p(B))$ with $\|\xi\|_{L_q(\tilde{l}_q(B^*))} \leq \|l\|$.

Lemma 4 Let B be a reflexive Banach space, $1 < p < +\infty$, $\xi = (\xi_1, \xi_2, \dots, \xi_n, \dots) \in L_p(\tilde{l}_p(B))$. Define the two-parameter martingale $\varphi = (\varphi_n, n \in \mathbb{N}^2)$ by

$$\varphi_0 = 0, \varphi_n = \sum_{i=1}^n [E_i(\xi_i) - E_{i-1}(\xi_i)], \forall n \geq 1. \quad (7)$$

Then $\varphi \in {}_p H_p^s(B)$ and $\|\varphi\|_{{}_p H_p^s(B)} \leq \|\xi\|_{L_p(\tilde{l}_p(B))}$.

Proof It is obvious that $\varphi = (\varphi_n, n \in \mathbb{N}^2)$ is a two-parameter B -valued martingale. From (7), for any

$i \geq 1$, we get

$$\|d_i \varphi\|^p = \|E_i(\xi_i) - E_{i-1}(\xi_i)\|^p,$$

and

$$E_{i-1}(\|d_i \varphi\|^p) \leq 2^{p-1} E_{i-1}[\|E_i(\xi_i)\|^p + \|E_{i-1}(\xi_i)\|^p] \leq 2^p E_{i-1}(\|\xi_i\|^p).$$

Therefore

$$s^{(p)}(\varphi) = \left[\sum_{i=1}^{\infty} E_{i-1}(\|d_i \varphi\|^p) \right]^{\frac{1}{p}} \leq \left[2^p \sum_{i=1}^{\infty} E_{i-1}(\|\xi_i\|^p) \right]^{\frac{1}{p}} = 2 \left[\sum_{i=1}^{\infty} E_{i-1}(\|\xi_i\|^p) \right]^{\frac{1}{p}}.$$

Hence,

$$\|\varphi\|_{{}_p H_p^s(B)} = \|s^{(p)}(\varphi)\|_{L_p(R)} \leq 2 \left[\sum_{i=1}^{\infty} E_{i-1}(\|\xi_i\|^p) \right]^{\frac{1}{p}} \| \xi \|_{L_p(\tilde{I}_p(B))},$$

which gives the desired result.

Lemma 4 is proved.

Theorem 2 Let B be a reflexive Banach space, $1 < p < +\infty$, then $({}_p H_p^s(B))^* \approx {}_q H_q^s(B^*)$, where $1/p + 1/q = 1$.

Proof For any $\phi \in {}_q H_q^s(B^*)$, we define

$$l_\phi(f) \triangleq \lim_{n \rightarrow \infty} E(f_n \cdot \phi_n), \forall f \in {}_p H_p^s(B).$$

By Hölder inequality, we have

$$\begin{aligned} |E(f_n \cdot \phi_n)| &= |E(\sum_{i=1}^n d_i f \cdot \sum_{i=1}^n d_i \phi)| = \\ &= |E(\sum_{i=1}^n d_i f \cdot d_i \phi)| = |E(\sum_{i=1}^n E_{i-1}(d_i f \cdot d_i \phi))| \leq \\ &= E[\sum_{i=1}^n (E_{i-1}(\|d_i f\|^p))^{\frac{1}{p}} \cdot (E_{i-1}(\|d_i \phi\|^q))^{\frac{1}{q}}] \leq \\ &= E[(\sum_{i=1}^n E_{i-1}(\|d_i f\|^p))^{\frac{1}{p}} \cdot (\sum_{i=1}^n E_{i-1}(\|d_i \phi\|^q))^{\frac{1}{q}}] = \\ &= E(s_n^{(p)}(f) s_n^{(q)}(\phi)) \leq \|s_n^{(p)}(f)\|_{L_p(R)} \cdot \|s_n^{(q)}(\phi)\|_{L_q(R)} \leq \\ &= \|f\|_{{}_p H_p^s(B)} \cdot \|\phi\|_{{}_q H_q^s(B^*)}. \end{aligned}$$

This shows that ${}_q H_q^s(B^*) \subset ({}_p H_p^s(B))^*$.

Next, we prove $({}_p H_p^s(B))^* \subset {}_q H_q^s(B^*)$. At first, employing lemma 4, we can consider $L_p(\tilde{I}_p(B))$ as a subspace of ${}_p H_p^s(B)$, which consists of $\theta = df = (d_1 f, d_2 f, \dots)$ for all martingale $f = (f_n, n \in \mathbb{N}^2) \in {}_p H_p^s(B)$, and we have $\|f\|_{{}_p H_p^s(B)} \leq 2 \|\theta\|_{L_p(\tilde{I}_p(B))}$. Now, for any $l(\cdot) \in ({}_p H_p^s(B))^*$, $l(\cdot)$ is also a bounded linear functional on $L_p(l_p(B))$. By lemma 3, there exists

some $\xi = (\xi_1, \xi_2, \dots) \in L_q(\tilde{I}_q(B^*))$, such that

$$l(f) = l_\xi(\theta) = \sum_{i=1}^{\infty} E(\xi_i \theta_i) = E(\sum_{i=1}^{\infty} \xi_i d_i f), \quad \forall f \in {}_p H_p^s(B), \quad (8)$$

and it satisfies that

$$\|\xi\|_{L_q(\tilde{I}_q(B^*))} \leq \|l\|.$$

Apply lemma 4 again and replacing B by B^* , set $\varphi = (\varphi_n, n \in \mathbb{N}^2)$ with $\varphi_0 = 0$, $\varphi_n = \sum_{i=1}^n [E_i(\xi_i) - E_{i-1}(\xi_i)]$, $\forall n \geq 1$. Then φ is a martingale in ${}_q H_q^s(B^*)$ and

$$\|\varphi\|_{{}_q H_q^s(B^*)} \leq 2 \|\xi\|_{L_q(\tilde{I}_q(B^*))}. \quad (9)$$

Combining (8) and (9), in other word, for any $l(\cdot) \in ({}_p H_p^s(B))^*$, there exists some martingale $\varphi \in {}_q H_q^s(B^*)$, such that

$$l(f) = l_\varphi(f) = E(\sum_{i=1}^{\infty} d_i f d_i \varphi), \quad \forall f \in {}_p H_p^s(B),$$

and $\|l\| \geq \|\varphi\|_{{}_q H_q^s(B^*)} / 2$, this means that $({}_p H_p^s(B))^* \subset {}_q H_q^s(B^*)$.

Theorem 2 is proved.

Theorem 3 Let B be a reflexive Banach space, $1 < p < +\infty$, $0 < \alpha \leq 1$, then $({}_p H_\alpha^s(B))^* \approx L_{q,\beta}^{s(q)}(B^*)$, where $q = p/(p-1)$ and $\beta = 1/\alpha - 1$.

Proof Firstly, by theorem 1, it has been proved that $L_{q,\beta}^{s(q)}(B^*) \subset ({}_p H_\alpha^s(B))^*$. Conversely, for any $l(\cdot) \in ({}_p H_\alpha^s(B))^*$, since ${}_p H_p^s(B) \subset {}_p H_\alpha^s(B)$, $l(\cdot)$ is also a bounded linear functional on ${}_p H_p^s(B)$. By theorem 2, $({}_p H_p^s(B))^* \approx {}_q H_q^s(B^*)$, then there exists a martingale $\phi \in {}_q H_q^s(B^*)$, such that

$$l(f) = E(f \phi) \triangleq \lim_{n \rightarrow \infty} E(f_n \phi_n), \quad \forall f \in {}_p H_p^s(B).$$

For any stopping time $\nu \in T^2$, set $b = \phi - \phi^\nu$. Then $b \in {}_q H_q^s(B^*)$.

On the other hand, by the reflexivity of B and applying theorem 2, we have $({}_q H_q^s(B^*))^* \approx {}_p H_p^s(B^{**}) \approx {}_p H_p^s(B)$. Hence, for any $\varepsilon > 0$, there exists some $a_\varepsilon \in {}_p H_p^s(B)$ with $\|a_\varepsilon\|_{{}_p H_p^s(B)} < 1 + \varepsilon$, such that $\|b\|_{{}_q H_q^s(B^*)} = \|b - b^\nu\|_{{}_q H_q^s(B^*)} = E((b - b^\nu)a_\varepsilon) = E(b(a_\varepsilon - a_\varepsilon^\nu))$. Particularly, there exists a martingale $a \in {}_p H_p^s(B)$ with $\|a\|_{{}_p H_p^s(B)} \leq 2$, such that $\|b\|_{{}_q H_q^s(B^*)} = \|E(b(a - a^\nu))\|$. Let

$$g = \frac{a - a^v}{4P(v \neq \infty)^{\frac{1}{\alpha} - \frac{1}{p}}}.$$

Then $s^{(p)}(g) = s^{(p)}(g)\chi(v \neq \infty)$. By Hölder inequality, we get

$$\begin{aligned} \|g\|_{pH_\alpha^s(B)} &= \frac{[E(s^{(p)}(a - a^v)^\alpha \cdot \chi(v \neq \infty))]^{\frac{1}{\alpha}}}{4P(v \neq \infty)^{\frac{1}{\alpha} - \frac{1}{p}}} \leq \\ &= \frac{[(E(s^{(p)}(a - a^v)^p))^{\frac{\alpha}{p}} \cdot P(v \neq \infty)^{1 - \frac{\alpha}{p}}]^{\frac{1}{\alpha}}}{4P(v \neq \infty)^{\frac{1}{\alpha} - \frac{1}{p}}} = \\ &= \frac{1}{4} [(E(s^{(p)}(a - a^v)^p))^{\frac{1}{p}}]^{\frac{1}{\alpha}} \leq \\ &= \frac{1}{4} (\|a\|_{pH_\alpha^s(B)} + \|a^v\|_{pH_\beta^s(B)}) \leq 1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|I\| &\geq |I(g)| = |E(g\phi)| = |E(g(\phi - \phi^v))| = \\ &= E\left(\frac{a - a^v}{4P(v \neq \infty)^{\frac{1}{\alpha} - \frac{1}{p}}} \cdot b\right) = \frac{1}{4} P(v \neq \infty)^{\frac{1}{\alpha} - \frac{1}{p}} \cdot \\ &= \frac{1}{4} P(v \neq \infty)^{\frac{1}{\alpha} - \frac{1}{p}} \|b\|_{qH_\beta^s(B^*)} = \\ &= \frac{1}{4} P(v \neq \infty)^{-\frac{1}{q} - (\frac{1}{\alpha} - 1)} \|s^{(q)}(\phi - \phi^v)\|_{L_q(R)} = \\ &= \frac{1}{4} \|\phi\|_{L_{q,\beta}^{s(q)}(B^*)}, \end{aligned}$$

where $\beta = 1/\alpha - 1$. So $({}_pH_\alpha^s(B))^* \subset L_{q,\beta}^{s(q)}(B^*)$.

Theorem 3 is proved.

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小指标情形下两参数 B 值鞅空间 ${}_pH_\alpha^s(B)$ 的对偶

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摘要: 通过对两参数 B 值鞅引入新型 Campanato 空间, 运用原子分解的经典技巧, 证明了在小指标 $0 < \alpha \leq 1$ 情形下, 由条件 p 均方函数定义的两参数 B 值鞅空间 ${}_pH_\alpha^s(B)$ 的对偶空间等价于 $L_{q,\beta}^{s(q)}(B^*)$, 所得结论推广了单参数 B 值鞅和两参数实值鞅的相关结论.

关键词: 两指标 B 值鞅空间; 对偶空间; 原子分解

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